On the Turán Numbers for Even Cycles

Rui Zhang, Yongqi Sun, and Yali Wu

***Abstract*—For integers $k\geq 2$ and $n\geq 2k+1$, let ex (n, C_{2k}) denote the maximum number of edges in a C_{2k}-free graph of order n, and EX (n, C_{2k}) denote the set of all graphs with $ex(n, C_{2k})$ edges. For k=3, it is well known that ex (n, C_{2k}) > $0.5338n^{1+1/k}$ for some large n. In this paper, we study the values of ex (n, C_{2k}) for $k\geq 3$ when n is small, their lower bounds were given based the three graphs without C_{2k}. The known result shows that it is the tight lower bound for k=3 and n=28, and we further conjecture that $ex(n, C_{2k})=(2k+1)(2k-1)(2k-2)/2$ for $n=4k^2-2k-2$ and $k\geq 4$.**

***Index Terms*—Extremal graph, even cycle, lower bounds, Turán numbers.**

I. Introduction

We consider only finite undirected graphs without loops or multiple edges. $V(G)$ and $E(G)$ denote the vertex set and edge set of graph G respectively. P_k is a path on k vertices and C_i is a cycle of length i. The length of the shortest cycle in G is referred to as the girth of G, denoted by $g(G)$. A block of a graph is maximal nonseparable subgraph. Let $EC(S)$ denote the edge set of the complete graph with vertex set S.

For integers $k\geq 2$ and $n\geq 2k+1$, let $ex(n, C_{2k})$ denote the maximum number of edges in a C_{2k}-free graph of order n, and $EX(n, C_{2k})$ denote the set of all graphs with $ex(n, C_{2k})$ edges. There is a conjecture of Erdős and Simonovits [1] that $ex(n, C_{2k})$ is asymptotically to $(1/2)n^{1+1/k}$ as n tends to infinity. Bollobás [2] proved that $ex(n, C_{2k})\sim(1/2)n^{1+1/k}$ for k=2. Lazebnik, Ustimenko and Woldar [3] studied certain families of C_{2k}-free graphs, and proved that $ex(n, C_{2k})\geq((k-1)k^{-(1+1/k)}+o(1))n^{1+1/k}$ for k=3 and 5. In paper [4], they further improved the lower bounds on $ex(n, C_{2k})$ to $(0.5 + o(1))\, n^{1+1/k}$ for k=3. Füredi, Naor and Verstraëte [5] proved that $ex(n, C_{2k})>0.5338n^{4/3}$, where $n=t^3+t^2+t+1+\lfloor(\sqrt{5}-2)(t^3+t^2+t+1)\rfloor$ and $t=2^{2\alpha+1}$ for α being a positive integer. Hence, the smallest n for them is 723. Yang and Rowlinson found the values of $ex(n, C_4)$ for $21\leq n\leq 31$ [6] and the values of $ex(n, C_4)$ for $6\leq n\leq 21$ [7] by a computer. They also determined the corresponding extremal graphs. Sun [8] *et al*. expanded the results for $ex(n, C_6)$ to $n\leq 26$.

In this paper, the lower bounds on $ex(n, C_{2k})$ for $k\geq 3$ are studied when n is small. We first construct some C_{2k}-free graphs of order $n_0=n_0(k)$ to obtain the lower bounds on $ex(n_0, C_{2k})$. Then, the lower bounds on $ex(n, C_{2k})$ for $n< n_0$ are obtained from them by deleting some vertices. The known results in [9] show that lower bounds of $ex(n, C_{2k})$ we obtained when $n=4k^2-2k-2$ are tight. For the sake of convenience, let σ denote the size of K_{2k-1} in the following sections, namely

$$\sigma=\frac{(2k-1)(2k-2)}{2}.$$

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The authors are with Beijing Jiaotong University, China (e-mail: yqsun@bjtu.edu.cn).

II. Construction and Proofs

According to the value of k, we divide n into three parts, $n\leq 4k^2-2k-2$, $4k^2-2k-2<n\leq 2k^3-3k-1$ and $n>2k^3-3k-1$. Then the lower bounds on $ex(n, C_{2k})$ are obtained respectively.

1) $n\leq 4k2-2k-2$

We will construct a C_{2k}-free graph F_1 of order $n=4k^2-2k-2$ as follows. Firstly, we construct a cycle of length $2(2k+1)$, and label the vertices of the cycle by v_1, u_1, v_2, u_2,..., v_{2k+1}, u_{2k+1} sequentially. Secondly, we divide the cycle into $2k+1$ copies of P_3 ($v_iu_iv_{i+1}$ for $1\leq i\leq 2k$ and $v_{2k+1}u_{2k+1}v_1$), and expand each P_3 to a K_{2k-1} by adding $2k-4$ vertices, where $x_{i,1}, x_{i,2},\ldots, x_{i,2k-4}$ are added to $v_iu_iv_{i+1}$ for $1\leq i\leq 2k$ and $x_{2k+1,1}, x_{2k+1,2},\ldots, x_{2k+1,2k-4}$ to $v_{2k+1}u_{2k+1}v_1$. Now the $2k+1$ copies of K_{2k-1} are connected end to end. Let $E_1=\cup_{i=1}^{2k} EC(\{v_i, u_i, v_{i+1}, x_{i,1}, x_{i,2},\ldots, x_{i,2k-4}\})\cup EC(\{v_{2k+1}, u_{2k+1}, v_1, x_{2k+1,1}, x_{2k+1,2},\ldots, x_{2k+1,2k-4}\})$, then $|E_1| = (2k + 1)\sigma$. Hence we have

$$V(F_1)=\{v_i, u_i\colon 1\leq i\leq 2k+1\}\cup\{x_{i,j}\colon 1\leq i\leq 2k+1, 1\leq j\leq 2k-4\},$$
$$E(F_1)=E_1.$$

Note that we construct F_1 from a cycle of length $2(2k+1)$, and expand each P_3 to a K_{2k-1}. Furthermore, F_1 are consisting of $2k+1$ blocks of order $2k-1$. So F_1 contains no C_{2k}. Hence we have

$$ex(n, C_{2k})>(2k + 1)\sigma,$$

where $n=4k^2-2k-2$. Taking k=5 as an example, the graph F_1 of order 88 is shown in Fig. 1. Hence we have $ex(88, C_{10})\geq 396$.

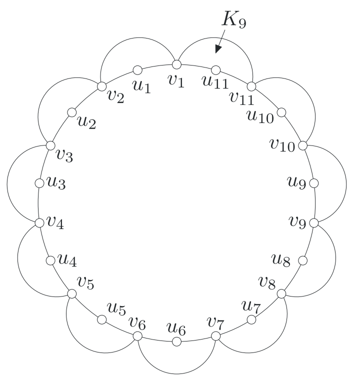

Fig. 1. A C_{10}-free graph of order 88.

Let n_0 denote the order of F_1. If $n\neq n_0$, we can construct the C_{2k}-free graphs of order n from F_1 by deleting vertices as follows. Let $\Delta n = n_0 - n$, the graph obtained from F_1 by deleting Δn vertices is denoted by $H_{\Delta n}$.

For $\Delta n \le 2k-4$, $H_{\Delta n}$ is obtained from F_1 by removing $x_{1,j}$ for $1 \le j \le \Delta n$. For $\Delta n = 2k-3$, $H_{\Delta n}$ is obtained from H_{2k-4} by removing u_1. Let $\alpha = \lfloor(\Delta n-(2k-3))/(2k-2)\rfloor$ and $\beta = (\Delta n-(2k-3)) \bmod (2k-2)$. For $\Delta n > 2k-3$, there are three cases according to the value of β. If $\beta=0$, $H_{\Delta n}$ is obtained from H_{2k-3} by removing $x_{i,j}$, u_i and v_i for $2 \le i \le \alpha+1$ and $1 \le j \le 2k-4$. If $1 \le \beta \le 2k-4$, $H_{\Delta n}$ is obtained from $H_{(2k-2)\alpha+2k-3}$ by removing $x_{\alpha+2,j}$ for $1 \le j \le \beta$. If $\beta = 2k-3$, $H_{\Delta n}$ is obtained from $H_{(2k-2)\alpha+2k-3+2k-4}$ by removing $u_{\alpha+2}$.

Since the graphs $H_{\Delta n}$ are obtained from F_1 by deleting Δn vertices, $H_{\Delta n}$ contains no C_{2k} for $\Delta n \ge n_0$. Hence we have the following lemma.

Lemma 1. For $n \le n_0 = 4k^2 - 2k - 2$, we have

$$ex(n;C_{2k}) \ge (2k+1)\sigma - \left[\frac{(4k-3-\beta)\beta}{2} + p(\sigma-1) + q(\alpha\sigma+1)\right], \quad (1)$$

where $\alpha = (\Delta n-(2k-3))/(2k-2)$ and $\Delta n = n_0 - n$. If $\Delta n \le 2k-3$, then $p=q=0$, and $\beta = \Delta n$. If $2k-3 < \Delta n \le 4k-6$, then $p=1$, $q=0$ and $\beta = \Delta n-(2k-3)$. If $\Delta n > 4k-6$, $p=q=1$ and $\beta = (\Delta n-(2k-3)) \bmod (2k-2)$.

Especially for $2k^2-3 < n \le 4k^2-2k-2$, we can use another method to construct C_{2k}-free graphs of order n from F_1 as follows. Let $\alpha = \lfloor \Delta n/(2k-2)\rfloor$ and $\beta = \Delta n \bmod (2k-2)$. There are three cases according to the value of β. If $\beta=0$, $H_{\Delta n}$ is obtained from F_1 by removing $x_{i,j}$, v_i and u_i, adding the edges $v_{\alpha+1}x_{2k+1,j}$, $v_{\alpha+1}v_{2k+1}$, $v_{\alpha+1}u_{2k+1}$ for $1 \le i \le \alpha$ and $1 \le j \le 2k-4$. Since $H_{\Delta n}$ contains no C_{2k}, we need to delete the edges $v_{2k+1}v_{2k}$, $v_{2k}v_{2k-1}, \ldots, v_{2k-\alpha+2}v_{2k-\alpha+1}$ from it. If $1 \le \beta \le 2k-4$, $H_{\Delta n}$ is obtained from $H_{(2k-2)\alpha}$ by removing $x_{\alpha+1,j}$ for $1 \le j \le \beta$. If $\beta = 2k-3$, $H_{\Delta n}$ is obtained from $H_{(2k-2)\alpha+2k-4}$ by removing $u_{\alpha+1}$. Since the graphs $H_{\Delta n}$ are obtained from F_1 by deleting Δn vertices, we have $H_{\Delta n}$ contains no C_{2k}. Hence we have the following lemma.

Lemma 2. For $2k^2-3 < n \le n_0 = 4k^2 - 2k - 2$, we have

$$ex(n;C_{2k}) \ge (2k+1)\sigma - \left[\frac{(4k-3-\beta)\beta}{2} + (\sigma+1)\alpha\right], \quad (2)$$

where $\alpha = \lfloor \Delta n/(2k-2)\rfloor$, $\beta = \Delta n \bmod (2k-2)$ and $\Delta n = n_0 - n$. By Lemma 1 and Lemma 2, we have the lower bounds on $ex(n, C_{2k})$ for $2k^2-3 < n \le n_0 = 4k^2 - 2k - 2$ as following.

Lemma 3. For $2k^2-3 < n \le n_0 = 4k^2 - 2k - 2$, we have

$$ex(n, C_{2k}) \ge (2k+1)\sigma - [\min\{\Omega_1, \Omega_2\}], \quad (3)$$

where the values of Ω_1 and Ω_2 are equal to the ones enclosed in square bracket of the inequalities (1) and (2) respectively.

2) $4k^2-2k-2 < n \le 2k^3-3k-1$

A C_{2k}-free graph F_2 of order $n=2k^3-3k-1$ is constructed as follows. Firstly, we construct a cycle of length $2(2k+2)$, and label the vertices of the cycle by $v_1, u_1, v_2, u_2, \ldots, v_{2k+2}, u_{2k+2}$ sequentially. Secondly, we divide the cycle into $2k+2$ copies of P_3 ($v_iu_iv_{i+1}$ for $1 \le i \le 2k+1$ and $v_{2k+2}u_{2k+2}v_1$), and expand each P_3 to a K_{2k-1} by adding $2k-4$ vertices, where $x_{i,1}, x_{i,2}, \ldots, x_{i,2k-4}$ are added to $v_iu_iv_{i+1}$ for $1 \le i \le 2k+1$ and $x_{2k+2,1}, x_{2k+2,2}, \ldots, x_{2k+2,2k-4}$ to $v_{2k+2}u_{2k+2}v_1$. Now the $2k+2$ copies of K_{2k-1} are connected end to end. Let $E_1 = \cup_{i=1}^{2k+1} EC(\{v_i, u_i, v_{i+1}, x_{i,1}, x_{i,2}, \ldots, x_{i,2k-4}\}) \cup EC(\{v_{2k+2}, u_{2k+2}, v_1, x_{2k+2,1}, x_{2k+2,2}, \ldots, x_{2k+2,2k-4}\})$, then $|E_1| = (2k+2)\sigma$. Thirdly, we connect u_i and u_{i+k+1} for $1 \le i \le k+1$, and divide them into $k-1$ parts by vertices $y_{i,1}, y_{i,2}, \ldots, y_{i,k-2}$. The P_3 $(u_i, y_{i,1}, y_{i,2})$ is expanded to a $K_{2k-1} - e(u_iy_{i,2} \in E(F_2))$ by adding vertices $z_{i,1,1}, z_{i,1,2}, \ldots, z_{i,1,2k-4}$, where $1 \le i \le k+1$. Each of the remaining $k-3$ parts is expanded to K_{2k-1} by adding $2k-3$ vertices as follows. The vertices $z_{i,j,1}, z_{i,j,2}, \ldots, z_{i,j,2k-3}$ are added to $y_{i,j}y_{i,j+1}$ for $2 \le j \le k-3$, and $z_{i,k-2,1}, z_{i,k-2,2}, \ldots, z_{i,k-2,2k-3}$ to $y_{i,k-2}u_{i+k+1}$, where $1 \le i \le k+1$. Let $E_2^i = \cup_{j=2}^{k-3} EC(\{y_{i,j}, y_{i,j+1}, z_{i,j,1}, z_{i,j,2}, \ldots, z_{i,j,2k-3}\}) \cup (EC(\{u_i, y_{i,1}, y_{i,2}, z_{i,1,1}, z_{i,1,2}, \ldots, z_{i,1,2k-4}\}) \backslash \{u_iy_{i,2}\}) \cup EC(\{u_{i+k+1}, y_{i,k-2}, z_{i,k-2,1}, z_{i,k-2,2}, \ldots, z_{i,k-2,2k-3}\})$, then $|E_2^i| = (k-2)\sigma - 1$. Hence we have

$$\begin{gathered} V(F_2) = \{v_i, u_i : 1 \le i \le 2k+2\} \cup \\ \{x_{i,j} : 1 \le i \le 2k+2,\ 1 \le j \le 2k-4\} \cup \\ \{y_{i,j} : 1 \le i \le k+1,\ 1 \le j \le k-2\} \cup \\ \{z_{i,j,l} : 1 \le i \le k+1,\ 1 \le j \le k-2,\ 1 \le l \le 2k-4 \text{ (for } j=1) \text{ or } 1 \le l \le 2k-3 \text{ (for } 1<j \le k-2)\}, \end{gathered}$$

$$E(F_2) = \cup_{i=1}^{k+1} E_2^i \cup E_1.$$

Then $|E(F_2)| = ((k-2)\sigma-1)\times(k+1) + (2k+2)\sigma = (k+1)(k\sigma-1)$. Note that F_2 is constructed from a graph G with girth $g(G) > 2k+1$. Furthermore, F_2 consists of $k(k+1)$ blocks of order $2k-1$. So F_2 contains no C_{2k}. Hence we have

$ex(n, C_{2k}) \ge (k+1)(k\sigma-1)$,

where $n=2k^2-3k-1$. Taking $k=5$ as an example, the graph F_2 of order 234 is shown in Fig. 2. Hence we have $ex(234, C_{10}) \ge 1074$.

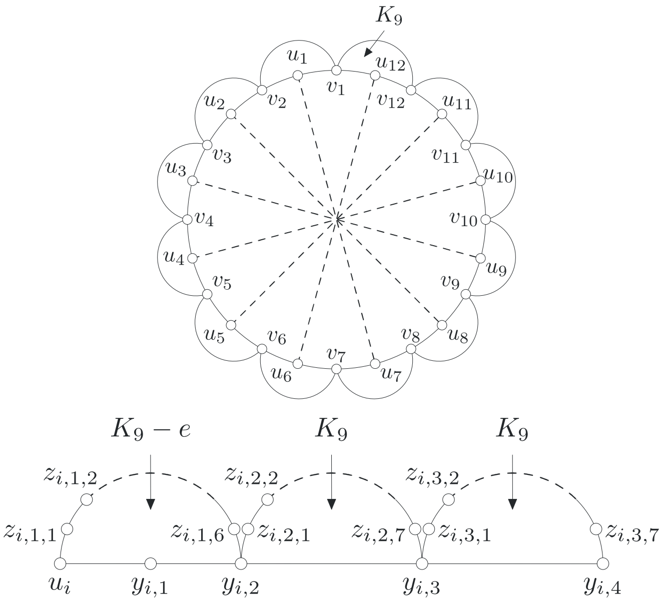

Fig. 2. The graph F_2 of order 234.

Let n_0 denote the order of the graph F_2. If $n=n_0$, we can construct the C_{2k}-free graphs of order n by deleting vertices from F_2 as follows. Let $\Delta n = n_0 - n$, the graph obtained from F_2 by deleting some vertices is denoted by $H_{\Delta n}$. Let $m = 2k^2-6k+3$ be the number of the vertices on each chord u_iu_{i+k+1} except u_i and u_{i+k+1} for $1 \le i \le k+1$. And let $\gamma = \lfloor \Delta n/m \rfloor$ and $\theta = \Delta n - \gamma m$. $H_{\gamma m}$ is obtained from F_2 by removing $z_{i,j,l}$ and $y_{i,j}$ for $1 \le i \le \gamma$, $1 \le j \le k-2$ and $1 \le l \le 2k-4$ ($j=1$) or $1 \le l \le 2k-3$ ($1<j \le k-2$).

For $\Delta n \le (k+1)m$, there are three subcases according to the value of θ. If $\theta \le 2k-4$, $H_{\Delta n}$ is obtained from $H_{\gamma m}$ by removing

$z_{\gamma+1,1,l}$ for $1\le l\le\theta$. If $\theta=2k-3$, $H_{\Delta n}$ is obtained from $H_{\gamma m+2k-4}$ by removing $y_{\gamma+1,1}$. There remains the case of $2k-3<\theta\le m-1$. Let $\alpha=\lfloor(\theta-(2k-3))/(2k-2)\rfloor$ and $\beta=(\theta-(2k-3)) \bmod (2k-2)$. $H_{\Delta n}$ is obtained from $H_{\gamma m+2k-3}$ by removing $z_{\gamma+1,j,l}$ and $y_{\gamma+1,j}$ for $2\le j\le\alpha+1$ and $1\le l\le 2k-3$, and removing $z_{\gamma+1,\alpha+2,l}$ for $1\le l\le\beta$. We can notice that the graph $H_{(k+1)m}$ is obtained from F_2 by deleting all the vertices on each chord u_iu_{i+k+1} except u_i and u_{i+k+1} for $1\le i\le k+1$.

For $\Delta n>(k+1)m$, $H_{\Delta n}$ is obtained from $H_{(k+1)m}$ by removing the vertices $x_{1,j}$ for $1\le j\le\theta$. Hence we have the following lemma.

Lemma 4. For $4k^2-2k-2< n\le n_0 =2k^2-3k-1$, we have

$$ex(n;C_{2k})\ge(k+1)(k\sigma-1)-\left[((k-2)\sigma-1)\gamma+\frac{(4k-3-\beta)\beta}{2}+p((\sigma-1)+\alpha\sigma)\right], \tag{4}$$

where $\alpha=\lfloor\Delta n-\gamma m-(2k-3)\rfloor/(2k-2)$, $\gamma=\lfloor\Delta n/m\rfloor$, $\Delta n=n_0-n$ and $m=2k^2-6k+3$. If $\Delta n-\gamma m\le 2k-3$, then $p=0$ and $\beta=\Delta n-\gamma m$, otherwise $p=1$ and $\beta=(\Delta n-(2k-3)(\gamma+1)) \bmod 2k-2$.

3) $n>2k^3-3k-1$

If $(n-2k^3+3k+1) \bmod (2k^2-2k-1)\equiv 0$, we will construct the C_{2k}-free graphs of order n denoted by F_3 as follows. Let $t=2n/(2k^2-2k-1)$. Firstly, we construct a cycle of length $2t$, and label the vertices of the cycle by $v_1, u_1, v_2, u_2,\ldots, v_t, u_t$ sequentially. Secondly, we divide the cycle into t copies of P_3 ($v_iu_iv_{i+1}$ for $1\le i\le t-1$ and $v_tu_tv_1$), and expand each P_3 to a K_{2k-1} by adding $2k-4$ vertices, where $x_{i,1}, x_{i,2},\ldots, x_{i,2k-4}$ are added to $v_iu_iv_{i+1}$ for $1\le i\le t-1$ and $x_{t,1}, x_{t,2}, \ldots, x_{t,2k-4}$ to $v_tu_tv_1$. Now the t copies of K_{2k-1} are connected end to end. Let $E_1=\cup_{i=1}^{t-1}EC(\{v_i, u_i, v_{i+1}, x_{i,1}, x_{i,2},\ldots, x_{i,2k-4}\})\cup EC(\{v_t, u_t, v_1, x_{t,1}, x_{t,2},\ldots, x_{t,2k-4}\})$, then $|E_1|=t\sigma$. Thirdly, we connect u_i and $u_{i+t/2}$ for $1\le i\le t/2$, and divide them into $k-1$ or $k-2$ parts according to the value of i as follows. For an odd i, $u_iu_{i+t/2}$ are divided into $k-1$ parts by vertices $y_{i,1}, y_{i,2},\ldots, y_{i,k-2}$. The P_3 ($u_iy_{i,1}y_{i,2}$) is expanded to a $K_{2k-1}-e$ ($u_iy_{i,2}\in E(F_3)$) by adding vertices $z_{i,1,1}, z_{i,1,2}, \ldots, z_{i,1,2k-4}$. Each of the remaining $k-3$ parts is expanded to K_{2k-1} by adding $2k-3$ vertices as follows. The vertices $z_{i,j,1}, z_{i,j,2}, \ldots, z_{i,j,2k-3}$ are added to $y_{i,j}y_{i,j+1}$ for $2\le j\le k-3$, and $z_{i,k-2,1}, z_{i,k-2,2}, \ldots, z_{i,k-2,2k-3}$ to $y_{i,k-2}u_{i+t/2}$. Let $E_2^i=\cup_{j=2}^{k-3}EC(\{y_{i,j}, y_{i,j+1}, z_{i,j,1}, z_{i,j,2}, \ldots, z_{i,j,2k-3}\})\cup(EC(\{u_i, y_{i,1}, y_{i,2}, z_{i,1,1}, z_{i,1,2}, \ldots, z_{i,1,2k-4}\})\backslash\{u_iy_{i,2}\})\cup EC(\{u_{i+t/2}, y_{i,k-2}, z_{i,k-2,1}, z_{i,k-2,2}, \ldots, z_{i,k-2,2k-3}\})$, then $|E_2^i|=(k-2)\sigma-1$. For an even i, $u_iu_{i+t/2}$ are divided into $k-2$ parts by vertices $y_{i,1}, y_{i,2},\ldots, y_{i,k-3}$, and each part is expanded to a K_{2k-1} by adding $2k-3$ vertices, where $z_{i,j,1}, z_{i,j,2},\ldots, z_{i,j,2k-3}$ are added to the j-th part for $1\le j\le k-2$. Let $E_2^i=\cup_{j=2}^{k-3}EC(\{y_{i,j-1}, y_{i,j}, z_{i,j,1}, z_{i,j,2},\ldots, z_{i,j,2k-3}\})\cup EC(\{u_i, y_{i,1}, z_{i,1,1}, z_{i,1,2}, \ldots, z_{i,1,2k-3}\})\cup EC(\{u_{i+t/2}, y_{i,k-3}, z_{i,k-2,1}, z_{i,k-2,2}, \ldots, z_{i,k-2,2k-3}\})$, then $|E_2^i|=(k-2)\sigma$. Hence we have

$$\begin{gathered}V(F_3)=\{v_i, u_i{:}1\le i\le t\}\cup\\ \{x_{i,j}{:}1\le i\le t,\ 1\le j\le 2k-4\}\cup\\ \{y_{i,j}{:}1\le i\le t/2,\ 1\le j\le k-3+(i \bmod 2)\}\cup\\ \{z_{i,j,l}{:}1\le i\le t/2,\ 1\le j\le k-2,\ 1\le l\le 2k-4\ (for\ j=1)\ or\ 1\le l\le\\ 2k-3\ (for\ 1<j\le k-2)\},\\ E(F_3)=\cup_{i=1}^{t/2}E_2^i\cup E_1.\end{gathered}$$

Then $|E(F_3)|=((k-2)\sigma\times t/2-\lceil t/4\rceil)+t\sigma=kt\sigma/2-\lceil t/4\rceil$. Note that F_3 is constructed from a graph G with girth $g(G)>2k+1$. Furthermore, F_3 consists of $kt/2$ blocks of order $2k-1$. So F_3 contains no C_{2k}. Hence we have

$$ex(n, C_{2k})>kt\sigma/2-\lceil t/4\rceil,$$

where $(n-2k^3+3k+1) \bmod (2k^2-2k-1)\equiv 0$. Taking $k=5$ as an example, the graph F_3 of order 273 is shown in Fig. 3. Hence we have $ex(273, C_{10})>1256$.

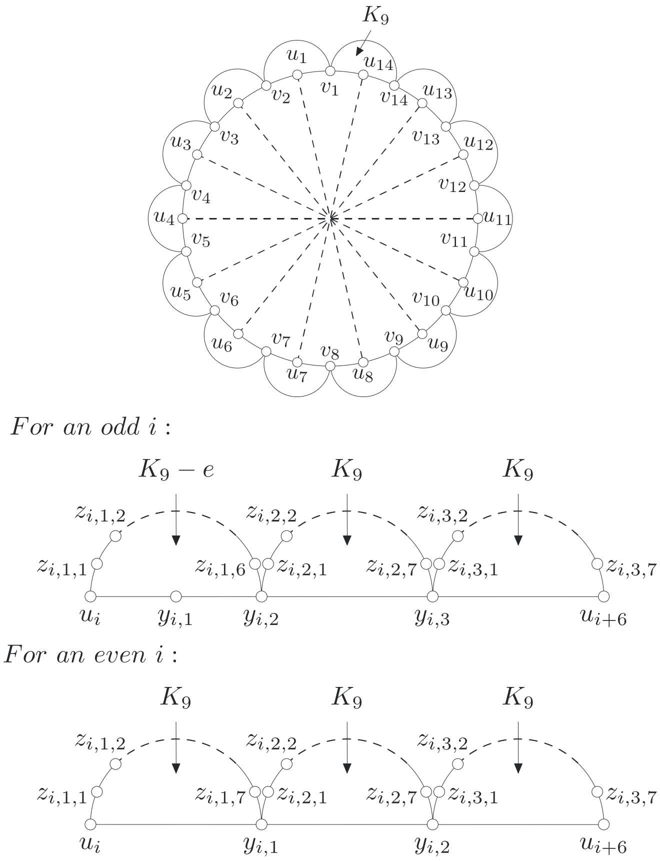

Fig. 3. The C_{10}-free graphs of order 273.

Let n_0 denote the order of the graph F_3. If $n=n_0$, we can construct the C_{2k}-free graphs of order n by deleting vertices from F_3 as follows, where $n_0-(2k^2-2k-2)\le n<n_0$. Let $\Delta n=n_0-n$, the graph obtained from F_3 by deleting some vertices is denoted by $H_{\Delta n}$.

For $\Delta n\le 2k-4$, $H_{\Delta n}$ is obtained from F_3 by removing the vertices $z_{1,1,l}$ for $1\le l\le\Delta n$. For $\Delta n=2k-3$, H_{2k-3} is obtained from H_{2k-4} by removing the vertex $y_{1,1}$. For $2k-3<\Delta n\le 2k^2-6k+3$, let $\alpha=\lfloor(\Delta n-(2k-3))/(2k-2)\rfloor$ and $\beta=(\Delta n-(2k-3)) \bmod (2k-2)$, then $H_{\Delta n}$ is obtained from H_{2k-3} by removing $z_{1,j,l}$ and $y_{1,j}$ for $2\le j\le\alpha+1$ and $1\le l\le 2k-3$. If $\beta=0$, we remove $z_{1,\alpha+2,l}$ for $1\le l\le\beta$ from $H_{2k-3+(2k-2)\alpha}$. We can notice that the graph H_{2k^2-6k+3} is obtained from F_3 by deleting all the vertices on each chord $u_1u_{1+t/2}$ except u_1 and $u_{1+t/2}$.

For $\Delta n>2k^2-6k+3$, we need to delete the vertices of two copies of K_{2k-1} including u_1 and $u_{1+t/2}$ respectively. The detail is as follows. For $2k^2-6k+3<i\le 2k^2-4k-1$, $H_{\Delta n}$ is obtained from H_{2k^2-6k+3} by removing the vertices $x_{1,j}$ for $1\le j\le\Delta n-(2k^2-6k+3)$. For $\Delta n=2k^2-4k$, $H_{\Delta n}$ is obtained from H_{2k^2-4k-1} by removing the vertex u_1. For $\Delta n=2k^2-4k+1$, $H_{\Delta n}$ is obtained from H_{2k^2-4k} by removing the vertex v_1, and adding the edges $v_2x_{t,j}$, v_2v_t and v_2u_t for $1\le j\le 2k-4$. For $2k^2-4k+1<\Delta n\le 2k^2-2k-3$, $H_{\Delta n}$ is obtained from H_{2k^2-4k+1} by removing the vertices $x_{1+t/2,j}$ for $1\le j\le\Delta n-(2k^2-4k+1)$. For $\Delta n=2k^2-2k-2$, $H_{\Delta n}$ is obtained from H_{2k^2-2k-3} by removing the vertex $u_{1+t/2}$. Hence we have the following lemma.

Lemma 5. Let $n_0>2k^3-3k$ and $n_0 \bmod (2k^2-2k-1)\equiv 0$. For $n_0-(2k^2-2k-2) \le n \le n_0$, we have

$$ex(n;C_{2k}) \ge \frac{kt\sigma}{2} - \left\lceil \frac{t}{4} \right\rceil - \left\lceil \frac{(4k-3-\beta)\beta}{2} + p((\sigma-1)+\alpha\sigma) \right\rceil, \quad (5)$$

where $\alpha=\lfloor(\Delta n-(2k-3))/(2k-2)\rfloor$, $t=2n_0/(2k^2-2k-1)$ and $\Delta n=n_0-n$. If $\Delta n\le 2k-3$ then $p=0$ and $\beta=\Delta n$, otherwise $p=1$ and $\beta=(\Delta n-(2k-3)) \bmod (2k-2)$.

III. Conclusion

By Lemma 1 and Lemma 3-5, we have the following theorem,

Theorem 6.

$$ex(n, C_{2k}) \ge \begin{cases} (2k+1)\sigma-M_1, & n\le 2k^2-3, \\ (2k+1)\sigma-M_2, & 2k^2-3<n\le 4k^2-2k-2, \\ (k+1)(k\sigma-1)-M_3, & 4k^2-2k-2<n\le 2k^3-3k-1, \\ kt\sigma/2-\lceil t/4 \rceil-M_4, & 2k^3-3k-1>n, \end{cases}$$

where $\sigma=(2k-1)(2k-2)/2$. If $n>2k^3-3k-1$, then $t=2n_0/(2k^2-3k-1)$, where $n\le n_0\le n+2k^2-2k-2$ and $n_0 \bmod (2k^2-2k-1) \equiv 0$. The values of $M_i (1\le i\le 4)$ are equal to the ones enclosed in square bracket of inequalities (1),(3)-(5) respectively.

For a graph G, let $d(G)=|E(G)|/|V(G)|$ denote the standard density. In Theorem 6, the lower bounds of $ex(n, C_{2k})$ are $(2k+1)(2k-1)(2k-2)/2$ when $n=4k^2-2k-2$. Let G be the graph from which we obtained above lower bounds (see part A in section II), then $d(G) = (2k-1)/2$ which reaches a local peak. Hence these graphs are more likely to be extremal graphs. In fact, it was shown that $ex(n,C_{2k})=70$ for $n=28$ and $k=3$ in [9]. Furthermore, we have the following conjecture,

Conjecture 7. If $n=4k^2-2k-2$ for $k \ge 4$, then

$$ex(n, C_{2k})=(2k+1)(2k-1)(2k-2)/2.$$

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Rui Zhang was born in 1984 in Shenyang, Liaoning, China. He received his master degree in mathematics from Beijing Jiaotong University in 2008. He is now a PhD candidate of computer science at School of Computer and Information Technology, Beijing Jiaotong University, China. His current research interests are graph theory and algorithmic optimization.

Yongqi Sun was born in 1969 in Luoyang, Henan, China. He received his PhD degree in computer science from Dalian University of Technology in 2006. Since 2014, he has been a professor in School of Computer and Information Technology, Beijing Jiaotong University, China. His research interests include graph theory, image processing, parallel computing and algorithmic optimization.

Yali Wu was born in 1980 in Taiyuan, Shanxi, China. She received her master degree in computer science from Taiyuan University of Technology in 2007. She is now a PhD candidate of computer science at School of Computer and Information Technology, Beijing Jiaotong University, China. Her current research interests are graph theory and algorithmic optimization.