Numerical Modelling of Influenza Model with Diffusion

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***Abstract*—The *SEIR* model for the transmission dynamics of influenza is extended by the addition of second-order space derivatives to enable the geographic spread of the disease in a population which has not been vaccinated against it. The resulting system of four reaction-diffusion equations is solved by a convergent finite-difference technique which is first-order accurate in time and second-order accurate in space. The resulting methods are analyzed for local truncation errors and stability. The numerical results verify that the proposed method is more competitive in terms of numerical stability than the standard finite-difference method.**

***Index Terms*—*SEIR* model, reaction-diffusion system, finite-difference method, initial condition, influenza, infectious diseases.**

I. Introduction

Mathematical models for epidemics are tools that can be used to predict the epidemic outbreak which obtained from computer programming to calculate the various parameters such as the speed of the spread of the virus, density of population, age structure, distance and frequency of travel of the population in the area. Influenza has caused more morbidity and mortality than all other respiratory diseases. There are annual seasonal epidemics that cause about 500,000 deaths worldwide each year. The analysis of the spread of influenza epidemics dates back to the 18th century. Since then different researchers developed these models on the basis of different type of anthropological parameters such as traffic patterns, contact patterns at work, in schools, at homes, at public places and the airline traffic (see, for instance, [1], [2]-[8]). These studies, however, gave no detail on the numerical method(s) used to solve the resulting nonlinear boundary-value problems (BVPS). This paper proposes an implicit finite difference method for solving the influenza epidemic model proposed in [9]. Although implicit by construction, the method can be implemented explicitly.

The model monitors four populations namely: the proportion of susceptible $S(x, t)$, exposed $E(x, t)$, infected $I(x, t)$ and recovered $R(x, t)$ individuals, respectively, at time t and distance x from the origin $(-L \le x \le L)$. The total population denoted $N(x,t)$ is given by $N(x, t) = S(x, t) + E(x, t) + I(x, t) + R(x, t)$. The influenza epidemic model (Massad *et al*. [10]) consists of the following equations, for all $t \ge 0$ and $-L \le x \le L$,

$$\frac{\partial S}{\partial t} = -\beta \frac{(E+I)}{N} S - \mu S + rN\left(1 - \frac{N}{K}\right) + d_1 \frac{\partial^2 S}{\partial x^2}, \tag{1}$$

$$\frac{\partial E}{\partial t} = \beta \frac{(E+I)}{N} S - \omega_1 E + d_2 \frac{\partial^2 E}{\partial x^2}, \tag{2}$$

$$\frac{\partial I}{\partial t} = \sigma E - \omega_2 I + d_3 \frac{\partial^2 I}{\partial x^2}, \tag{3}$$

$$\frac{\partial R}{\partial t} = \kappa E + \gamma I - \mu R + d_4 \frac{\partial^2 R}{\partial x^2} \tag{4}$$

where $\omega_1 = (\mu + \sigma + \kappa)$, $\omega_2 = (\mu + \alpha + \gamma)$, and $d_1, \ldots, d_4$ are diffusion rates. For biological meaning, the parameters $\beta, \mu, r, K, \sigma, \kappa, \alpha$ and γ denote infection rate, natural mortality rate, birth rate, carrying capacity, duration of latency, recovery rate of latent, flu induced mortality rate and recovery rate of clinically ill, respectively.

The initial conditions [11] are

$$S(x,0) = S_0,\ E(x,0) = E_0,\ I(x,0) = I_0,\ R(x,0) = R_0; \quad -L \le x \le L, \tag{5}$$

and the boundary conditions used for the domain are

$$\frac{\partial S(-L,t)}{\partial x} = \frac{\partial E(-L,t)}{\partial x} = \frac{\partial I(-L,t)}{\partial x} = \frac{\partial R(-L,t)}{\partial x} = 0;\ t \ge 0, \tag{6}$$

$$\frac{\partial S(2,t)}{\partial x} = \frac{\partial E(2,t)}{\partial x} = \frac{\partial I(2,t)}{\partial x} = \frac{\partial R(2,t)}{\partial x} = 0;\ t \ge 0. \tag{6}$$

II. Numerical Methods

A. Discretization and Notations

A solution of the system (1)–(6) may be computed by finite-difference methods by discretizing the space interval $[-L, L]$ into M sub-intervals each of width $h > 0$, and the time interval $t \ge 0$ is discretized in steps each of length $\ell > 0$. The open region $\Omega = [-L, L] \times [t > 0]$ and its boundary $\partial\Omega$ consisting of the lines $x = -L$, $x = L$ and $t = 0$ are thus covered by a rectangular mesh, the mesh points having coordinates of the form $(x_m, t_n) = (mh, n\ell)$ where $x_m = -L + mh (m = 0, 1, 2, ..., M)$ and $t_n = n\ell (n = 0, 1, 2, ...)$. The solutions of (1)–(6) at the typical mesh point (x_m, t_n) are, of course, $S(x_m, t_n), E(x_m, t_n), I(x_m, t_n)$ and $R(x_m, t_n)$ which will be denoted by S_m^n, E_m^n, I_m^n, and R_m^n, respectively. A family of numerical methods will be developed by approximating the time derivatives in (1)–(4) by the first order

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forward-difference replacement

$$\frac{\partial u(x,t)}{\partial t}=\frac{u(x,t+\ell)-u(x,t)}{\ell}+O(\ell)\ \text{ as } \ell\to 0, \tag{7}$$

and the space derivatives in (1)–(4) by the weighted approximant

$$\frac{\partial^2 u(x,t)}{\partial x^2}\approx\frac{\theta\{u(x-h,t+\ell)-2u(x,t+\ell)+u(x+h,t+\ell)\}}{h^2}+\frac{(1-\theta)\{u(x-h,t)-2u(x,t)+u(x+h,t)\}}{h^2}, \tag{8}$$

in which $u(x,t)$ represents $S(x_m,t_n), E(x_m,t_n), I(x_m,t_n)$ or $R(x_m,t_n)$ and $0\le\theta\le 1$ is a parameter. When $\theta=0$, (8) is $O(h^2)$ as $h,\ell\to 0$ and is $O(h^2+\ell)$ as $h,\ell\to 0$ otherwise.

B. Development of Finite-Difference Method

A competitive finite-difference method for solving *S, E, I, R* in (1)–(6), based on approximating the time derivative in (7) and the space derivative in (8) and making appropriate approximations for the right hand side, are given by

$$\begin{aligned}\frac{S_m^{n+1}-S_m^n}{\ell}&=-\beta\frac{\left(E_m^n+I_m^n\right)}{N_m^n}S_m^{n+1}-\mu S_m^{n+1}\\&+rN_m^n\left(1-\frac{N_m^n}{K}\right)+d_1\theta\left(\frac{S_{m-1}^{n+1}-2S_m^{n+1}+S_{m+1}^{n+1}}{h^2}\right)\\&+d_1(1-\theta)\left(\frac{S_{m-1}^n-2S_m^n+S_{m+1}^n}{h^2}\right),\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{E_m^{n+1}-E_m^n}{\ell}&=\beta\frac{\left(E_m^{n+1}+I_m^n\right)}{N_m^n}S_m^n-(\mu+\sigma+\kappa)E_m^{n+1}\\&+d_2\theta\left(\frac{E_{m-1}^{n+1}-2E_m^{n+1}+E_{m+1}^{n+1}}{h^2}\right)\\&+d_2(1-\theta)\left(\frac{E_{m-1}^n-2E_m^n+E_{m+1}^n}{h^2}\right),\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{I_m^{n+1}-I_m^n}{\ell}&=\sigma E_m^n-(\mu+\alpha+\gamma)I_m^{n+1}\\&+d_3\theta\left(\frac{I_{m-1}^{n+1}-2I_m^{n+1}+I_{m+1}^{n+1}}{h^2}\right)\\&+d_3(1-\theta)\left(\frac{I_{m-1}^n-2I_m^n+I_{m+1}^n}{h^2}\right),\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{R_m^{n+1}-R_m^n}{\ell}&=\kappa E_m^n+\gamma I_m^n-\mu R_m^{n+1}\\&+d_4\theta\left(\frac{R_{m-1}^{n+1}-2R_m^{n+1}+R_{m+1}^{n+1}}{h^2}\right)\\&+d_4(1-\theta)\left(\frac{R_{m-1}^n-2R_m^n+R_{m+1}^n}{h^2}\right),\end{aligned} \tag{12}$$

Equations (9)–(12) may be re-arranged to give, for $p=\ell/h^2$, $m=0,1,2,\dots,M$, $n=1,2,\dots,$

$$\begin{aligned}&-d_1p\theta S_{m-1}^{n+1}+d_SS_m^{n+1}-d_1p\theta S_{m+1}^{n+1}=d_1p(1-\theta)S_{m-1}^n\\&+\left[1-2d_1p(1-\theta)\right]S_m^n+d_1(1-\theta)S_{m+1}^n+\ell rA_m^n\end{aligned} \tag{13}$$

$$\begin{aligned}&-d_2p\theta E_{m-1}^{n+1}+d_EE_m^{n+1}-d_2p\theta E_{m+1}^{n+1}=d_2p(1-\theta)E_{m-1}^n\\&+\left[1-2d_2p(1-\theta)\right]E_m^n+d_2(1-\theta)E_{m+1}^n+\ell\beta I_m^nC_m^n\end{aligned} \tag{14}$$

$$\begin{aligned}&-d_3p\theta I_{m-1}^{n+1}+d_II_m^{n+1}-d_3p\theta I_{m+1}^{n+1}=d_3p(1-\theta)I_{m-1}^n\\&+\left[1-2d_3p(1-\theta)\right]I_m^n+d_3(1-\theta)I_{m+1}^n+\ell\sigma E_m^n,\end{aligned} \tag{15}$$

$$\begin{aligned}&-d_4p\theta R_{m-1}^{n+1}+d_RR_m^{n+1}-d_4p\theta R_{m+1}^{n+1}=d_4p(1-\theta)R_{m-1}^n\\&+\left[1-2d_4p(1-\theta)\right]R_m^n+d_4(1-\theta)R_{m+1}^n+\ell\kappa E_m^n+\ell\gamma I_m^n,\end{aligned} \tag{16}$$

where $A_m^n=N_m^n\left(1-\frac{N_m^n}{K}\right), B_m^n=\frac{E_m^n+I_m^n}{N_m^n}, C_m^n=\frac{S_m^n}{N_m^n}$,

$d_S=1+2d_1p\theta+\ell\mu+\ell\beta B_m^n,\ d_E=1+2d_2p\theta+\ell\omega_1-\ell\beta C_m^n,$
$d_I=1+2d_3p\theta+\ell\omega_2,\ d_R=1+2d_4p\theta+\ell\mu.$

The method (13)–(16) is denoted as method $DM(\theta)$. Verification of accuracy may be obtained by considering the local truncation errors,

$$\mathscr{L}_S=\mathscr{L}_S\left[S(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right],$$

$$\mathscr{L}_E=\mathscr{L}_E\left[S(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right],$$

$$\mathscr{L}_I=\mathscr{L}_I\left[S(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right],$$

and $\mathscr{L}_R=\mathscr{L}_R\left[S(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right]$,

associated with (13)–(16) at the point $(x,t)=(x_m,t_n)$, respectively, which may be obtained from (9)–(12). Using Taylor series expansion in (9)–(12) lead to

$$\begin{aligned}\mathscr{L}_S&=\frac{-1}{12}d_1h^2\frac{\partial^4S}{\partial x^4}\\&+\ell\left[\frac{1}{2}\frac{\partial^2S}{\partial t^2}+\left(\beta\frac{(E+I)}{N}+\mu\right)\frac{\partial S}{\partial t}-d_1\theta\frac{\partial^3S}{\partial x^2\partial t}\right]+\cdots,\end{aligned} \tag{17}$$

$$\begin{aligned}\mathscr{L}_E&=\frac{-1}{12}d_2h^2\frac{\partial^4E}{\partial x^4}\\&+\ell\left[\frac{1}{2}\frac{\partial^2E}{\partial t^2}+\omega_1\frac{\partial E}{\partial t}-d_2\theta\frac{\partial^3E}{\partial x^2\partial t}\right]+\cdots,\end{aligned} \tag{18}$$

$$\mathscr{L}_I=\frac{-1}{12}d_3h^2\frac{\partial^4I}{\partial x^4}+\ell\left[\frac{1}{2}\frac{\partial^2I}{\partial t^2}+\omega_2\frac{\partial I}{\partial t}-d_3\theta\frac{\partial^3I}{\partial x^2\partial t}\right]+\cdots, \tag{19}$$

$$\mathscr{L}_R=\frac{-1}{12}d_4h^2\frac{\partial^4R}{\partial x^4}+\ell\left[\frac{1}{2}\frac{\partial^2R}{\partial t^2}-\mu\frac{\partial R}{\partial t}-d_4\theta\frac{\partial^3R}{\partial x^2\partial t}\right]+\cdots, \tag{20}$$

Equations (17)–(20) verify that method $DM(\theta)$ is $O(h^2+\ell)$
as $h,\ell\to 0$.

C. Standard Finite-Difference Method

The corresponding standard finite-difference method for solving the initial/boundary value problems (IBVP) obtained by using (7) and (8) in (1)–(4) and evaluating the right-hand functions in (1)–(4) at base time level $t = t_n$, are given by, respectively, for $p = \ell / h^2,\ m = 0, 1, 2, ..., M$ and $n = 0, 1, 2, ...$

$$-d_1 p\theta S_{m-1}^{n+1} + (1+2d_1 p\theta)S_m^{n+1} - d_1 p\theta S_{m+1}^{n+1} = d_1 p(1-\theta)S_{m-1}^n + \left(d_{SS} - \ell\beta B_m^n\right)S_m^n + d_1(1-\theta)S_{m+1}^n + \ell r A_m^n, \quad (21)$$

$$-d_2 p\theta E_{m-1}^{n+1} + (1+2d_2 p\theta)E_m^{n+1} - d_2 p\theta E_{m+1}^{n+1} = d_2 p(1-\theta)E_{m-1}^n + \left(d_{EE} + \ell\beta C_m^n\right)E_m^n + d_2(1-\theta)E_{m+1}^n + \ell\beta I_m^n C_m^n, \quad (22)$$

$$-d_3 p\theta I_{m-1}^{n+1} + (1+2d_3 p\theta)I_m^{n+1} - d_3 p\theta I_{m+1}^{n+1} = d_3 p(1-\theta)I_{m-1}^n + d_{EE} I_m^n + d_3(1-\theta)I_{m+1}^n + \ell\sigma E_m^n, \quad (23)$$

$$-d_4 p\theta R_{m-1}^{n+1} + (1+2d_4 p\theta)R_m^{n+1} - d_4 p\theta R_{m+1}^{n+1} = d_4 p(1-\theta)R_{m-1}^n + d_{RR} R_m^n + d_4(1-\theta)R_{m+1}^n + \ell\kappa E_m^n + \ell\gamma I_m^n, \quad (24)$$

where

$$d_{SS} = 1 - \ell\mu - 2d_1 p(1-\theta),\ d_{EE} = 1 - \ell\omega_1 - 2d_2 p(1-\theta),$$
$$d_{II} = 1 - \ell\omega_2 - 2d_3 p(1-\theta),\ d_{RR} = 1 - \ell\mu - 2d_4 p(1-\theta),$$

The method (21)–(24) is denoted as method $SM(\theta)$.

Using Taylor series expansion, the associated local truncation errors at the point $(x, t) = (x_m, t_n)$ of the method $SM(\theta)$ are, respective,

$$\mathscr{L}_S = \frac{-1}{12} d_1 h^2 \frac{\partial^4 S}{\partial x^4} + \ell\left[\frac{1}{2}\frac{\partial^2 S}{\partial t^2} - d_1\theta\frac{\partial^3 S}{\partial x^2 \partial t}\right] + \cdots, \quad (25)$$

$$\mathscr{L}_E = \frac{-1}{12} d_2 h^2 \frac{\partial^4 E}{\partial x^4} + \ell\left[\frac{1}{2}\frac{\partial^2 E}{\partial t^2} - d_2\theta\frac{\partial^3 E}{\partial x^2 \partial t}\right] + \cdots, \quad (26)$$

$$\mathscr{L}_I = \frac{-1}{12} d_3 h^2 \frac{\partial^4 I}{\partial x^4} + \ell\left[\frac{1}{2}\frac{\partial^2 I}{\partial t^2} - d_3\theta\frac{\partial^3 I}{\partial x^2 \partial t}\right] + \cdots, \quad (27)$$

$$\mathscr{L}_R = \frac{-1}{12} d_4 h^2 \frac{\partial^4 R}{\partial x^4} + \ell\left[\frac{1}{2}\frac{\partial^2 R}{\partial t^2} - d_4\theta\frac{\partial^3 R}{\partial x^2 \partial t}\right] + \cdots, \quad (28)$$

Equations (25)–(28) verify that method $SM(\theta)$ is $O(h^2 + \ell)$ as $h, \ell \to 0$.

III. Stability Analyses

The von Neumann or Fourier series method seek the condition(s) under which small errors of the forms,

$$Z_{S,m}^n := S_m^n - \tilde{S}_m^n = e^{\psi_1 n\ell} e^{i\delta_1 mh}, \quad (29)$$

$$Z_{E,m}^n := E_m^n - \tilde{E}_m^n = e^{\psi_2 n\ell} e^{i\delta_2 mh}, \quad (30)$$

$$Z_{I,m}^n := I_m^n - \tilde{I}_m^n = e^{\psi_3 n\ell} e^{i\delta_3 mh}, \quad (31)$$

$$Z_{R,m}^n := R_m^n - \tilde{R}_m^n = e^{\psi_4 n\ell} e^{i\delta_4 mh}, \quad (32)$$

where $\psi_1, \psi_2, \psi_3, \psi_4, \delta_1, \delta_2, \delta_3$ and δ_4 are real, $i = \sqrt{-1}$ and $\tilde{S}_m, \tilde{E}_m, \tilde{I}_m, \tilde{R}_m$ are a perturbed numerical solutions. The necessary conditions for the errors not to grow as $n \to \infty$ are

$$\left|e^{\psi_1\ell}\right| \le 1 + M_S\ell,\ \left|e^{\psi_2\ell}\right| \le 1 + M_E\ell,\ \left|e^{\psi_3\ell}\right| \le 1 + M_I\ell,\ \left|e^{\psi_4\ell}\right| \le 1 + M_R\ell \quad (33)$$

where M_S, M_E, M_I, M_R are non-negative constants independent of h and ℓ. The conditions in (33) make no allowance for growing solutions if $M_S = 0, M_E = 0, M_I = 0$ and $M_R = 0$.

Method: $DM(\theta)$: Substituting Z_S into (21) leads to the (local) stability equation

$$\left\{1 + \ell\mu + \ell\beta B_m^n + 4d_1 p\theta \sin^2\frac{\delta_1 h}{2}\right\}\zeta_S = 1 - 4d_1 p(1-\theta)\sin^2\frac{\delta_1 h}{2} \quad (34)$$

where $\xi_S = e^{\psi_1\ell}$ and B_m^n are treated as a (local) constant. The von Neumann necessary condition for stability is $|\xi_S| < 1$, that is, the stability restrictions are

$$0 \le \theta < 1/2, \quad p \le \frac{2 + \ell\mu + \ell\beta B_m^n}{4d_1(1-2\theta)},$$
$$\theta = 1/2, \quad -\ell\mu - \ell\beta B_m^n \le 2, \quad p > 0, \quad (35)$$
$$1/2 < \theta \le 1, \quad p \ge \frac{-\ell\mu - \ell\beta B_m^n - 2}{4d_1(2\theta - 1)}.$$

Substituting Z_E into (22) gives the (local) stability equation

$$\left\{1 + \ell\omega_1 + 4d_2 p\theta \sin^2\frac{\delta_2 h}{2}\right\}\zeta_E = 1 + \ell\beta C_m^n - 4d_2 p(1-\theta)\sin^2\frac{\delta_2 h}{2}, \quad (36)$$

where $\xi_E = e^{\psi_2\ell}$ and C_m^n is treated as a (local) constant. The von Neumann necessary condition for stability is $|\xi_E| < 1$, that is, the stability restrictions are

$$0 \le \theta < 1/2,\ p \le \frac{2 + \ell\omega_1 + \ell\beta C_m^n}{4d_2(1-2\theta)} \text{ and } p \ge \frac{-\ell\omega_1 + \ell\beta C_m^n}{4d_2},$$

$$\theta = 1/2,\ 2 + \ell\omega_1 + \ell\beta C_m^n \ge 0 \text{ and } p \ge \frac{-\ell\omega_1 + \ell\beta C_m^n}{4d_2} \quad (37)$$

$$1/2 < \theta \le 1,\ p \le \frac{-2 - \ell\omega_1 - \ell\beta C_m^n}{4d_2(2\theta - 1)} \text{ and } p \ge \frac{-\ell\omega_1 + \ell\beta C_m^n}{4d_2}.$$

Substituting Z_I into (23) gives the (local) stability equation

$$\left\{1 + \ell\omega_2 + 4d_3 p\theta \sin^2\frac{\delta_3 h}{2}\right\}\zeta_I = 1 - 4d_3 p(1-\theta)\sin^2\frac{\delta_3 h}{2}, \quad (38)$$

where $\xi_I = e^{\psi_3 \ell}$. The von Neumann necessary condition for stability is $|\xi_I| < 1$, that is, the stability restrictions are

$$\begin{aligned} &0 \le \theta < 1/2, \quad p \le \frac{2+\ell\omega_2}{4d_3(1-2\theta)}, \\ &\theta = 1/2, \quad 2+\ell\omega_2 \ge 0, \;\; p > 0, \\ &1/2 < \theta \le 1, \quad p \le \frac{-2-\ell\omega_2}{4d_3(2\theta-1)}. \end{aligned} \tag{39}$$

Substituting Z_R into (24) gives the (local) stability equation

$$\left\{1+\ell\mu+4d_4 p\theta \sin^2\frac{\delta_4 h}{2}\right\}\zeta_R = 1-4d_4 p(1-\theta)\sin^2\frac{\delta_4 h}{2}, \tag{40}$$

where $\xi_R = e^{\psi_4 \ell}$. The von Neumann necessary condition for stability is $|\xi_R| < 1$, that is, the stability restrictions are

$$\begin{aligned} &0 \le \theta < 1/2, \quad p \le \frac{2-\ell\mu}{4d_4(1-2\theta)}, \\ &\theta = 1/2, \quad 2-\ell\mu \ge 0, \;\; p > 0, \\ &1/2 < \theta \le 1, \quad p \le \frac{\ell\mu-2}{4d_4(2\theta-1)}. \end{aligned} \tag{41}$$

Method: $SM(\theta)$: Similarly, substituting Z_S, Z_E, Z_I, Z_R into (21),(22),(23) and (24), respectively, yield the von Neumann stability restrictions as follows.

The stability restrictions of the method for S in (13) are

$$\begin{aligned} &0 \le \theta < 1/2, \;\; p \le \frac{2-\ell\mu-\ell\beta B_m^n}{4d_1(1-2\theta)}, \\ &\theta = 1/2, \;\; \ell\mu+\ell\beta B_m^n \le 2, \quad p > 0, \\ &1/2 < \theta \le 1, \;\; p \ge \frac{\ell\mu+\ell\beta B_m^n+2}{4d_1(2\theta-1)}. \end{aligned} \tag{42}$$

The stability restrictions of the method for E in (14) are

$$\begin{aligned} &0 \le \theta < 1/2, \;\; p \le \frac{2-\ell\omega_1+\ell\beta C_m^n}{4d_2(1-2\theta)} \text{ and } p \ge \frac{-\ell\omega_1+\ell\beta C_m^n}{4d_2}, \\ &\theta = 1/2, \;\; 2-\ell\omega_1+\ell\beta C_m^n \ge 0 \text{ and } p \ge \frac{-\ell\omega_1+\ell\beta C_m^n}{4d_2}, \\ &1/2 < \theta \le 1, \;\; p \le \frac{-2+\ell\omega_1-\ell\beta C_m^n}{4d_2(2\theta-1)} \text{ and } p \ge \frac{-\ell\omega_1+\ell\beta C_m^n}{4d_2}. \end{aligned} \tag{43}$$

The stability restrictions of the method for I (15) are

$$\begin{aligned} &0 \le \theta < 1/2, \;\; p \le \frac{2-\ell\omega_2}{4d_3(1-2\theta)}, \\ &\theta = 1/2, \;\; 2-\ell\omega_2 \ge 0, \;\; p > 0, \\ &1/2 < \theta \le 1, \;\; p \le \frac{-2+\ell\omega_2}{4d_3(2\theta-1)}. \end{aligned} \tag{44}$$

The stability restrictions of the method for R (16) are

$$\begin{aligned} &0 \le \theta < 1/2, \;\; p \le \frac{2-\ell\mu}{4d_4(1-2\theta)}, \\ &\theta = 1/2, \;\; 2-\ell\mu \ge 0, \quad p > 0, \\ &1/2 < \theta \le 1, \;\; p \le \frac{-2+\ell\mu}{4d_4(2\theta-1)}. \end{aligned} \tag{45}$$

It is followed that the stability intervals of the method $SM(\theta)$ and $DM(\theta)$ are given by the intersection of the restriction on p in each value of h, see Equations (42), (43), (44) and (45) and (35), (37), (39) and (41), respectively. These result show that the method $DM(\theta)$ is seen to be more competitive (in terms of stability intervals) than the method $SM(\theta)$. Numerical simulations is demonstrated in Section III to confirm these results.

IV. Implementation

The derivative $\frac{\partial S}{\partial x}$ in (6)-(7) may be approximated by the second-order, central-difference replacement

$$\frac{\partial S(x,t)}{\partial x} = \frac{S(x+h,t)-S(x-h,t)}{2h} + O(h^2), \tag{46}$$

with a similar replacement being made to $\frac{\partial E}{\partial x}$, $\frac{\partial I}{\partial x}$ and $\frac{\partial R}{\partial x}$. The implementation of the boundary conditions (6)-(7) to second-order, yield, on $x=-L$ and $x=L$, for $n=0,1,2,\ldots$

$$\begin{aligned} &S_1^n = S_{-1}^n,\; E_1^n = E_{-1}^n,\; I_1^n = I_{-1}^n,\; R_1^n = R_{-1}^n \text{ and} \\ &S_{M+1}^n = S_{M-1}^n,\; E_{M+1}^n = E_{M-1}^n,\; I_{M+1}^n = I_{M-1}^n,\; R_{M+1}^n = R_{M-1}^n. \end{aligned} \tag{47}$$

Thus, equation (47) introduce the exterior grid points $(x_{-1},t) = (-h, n\ell)$ and $(x_{M+1},t) = ((M+1)h, n\ell)$. let

$$\mathbf{S}^{n+1} = \left[S_0^{n+1}, S_1^{n+1}, \ldots, S_M^{n+1}\right]^T,\; \mathbf{E}^{n+1} = \left[E_0^{n+1}, E_1^{n+1}, \ldots, E_M^{n+1}\right]^T,$$
$$\mathbf{I}^{n+1} = \left[I_0^{n+1}, I_1^{n+1}, \ldots, I_M^{n+1}\right]^T \text{ and } \mathbf{R}^{n+1} = \left[R_0^{n+1}, R_1^{n+1}, \ldots, R_M^{n+1}\right]^T$$

where T denotes transpose. The implementation of methods $SM(\theta)$ and $DM(\theta)$, are as follows.

Method: $SM(\theta)$:

Taking $m = 0, M$ in (21)–(24) and using (47) gives $m = 0$,

$$\begin{aligned} (1+2d_1 p\theta)S_0^{n+1} - 2d_1 p\theta S_1^{n+1} &= \left(d_{SSS} - \ell\beta B_0^n\right)S_0^n \\ &+ 2d_1(1-\theta)S_1^n + \ell r A_0^n, \end{aligned} \tag{48}$$

$$\begin{aligned} (1+2d_2 p\theta)E_0^{n+1} - 2d_2 p\theta E_1^{n+1} &= \left(d_{EEE} + \ell\beta C_0^n\right)E_0^n \\ &+ 2d_2(1-\theta)E_1^n + \ell\beta I_0^n C_0^n, \end{aligned} \tag{49}$$

$$\begin{aligned} (1+2d_3 p\theta)I_0^{n+1} - 2d_3 p\theta I_1^{n+1} &= d_{III} I_0^n + 2d_3 p(1-\theta)I_1^n \\ &+ \ell\sigma E_0^n, \end{aligned} \tag{50}$$

$$\begin{aligned} (1+2d_4 p\theta)R_0^{n+1} - 2d_4 p\theta R_1^{n+1} &= d_{RRR} R_0^n + 2d_4(1-\theta)R_1^n \\ &+ \ell\kappa E_0^n + \ell\gamma I_0^n, \end{aligned} \tag{51}$$

and $m = M$,

$$\begin{aligned} -2d_1 p\theta S_{M-1}^{n+1} + (1+2d_1 p\theta)S_M^{n+1} &= 2d_1(1-\theta)S_{M-1}^n \\ &+ \left(d_{SSS} - \ell\beta B_M^n\right)S_M^n, + \ell r A_M^n, \end{aligned} \tag{52}$$

$$-2d_2 p\theta E_{M-1}^{n+1} + (1+2d_2 p\theta)E_M^{n+1} = 2d_2(1-\theta)E_{M-1}^n + \left(d_{EEE} + \ell\beta C_M^n\right)E_M^n + \ell\beta I_M^n C_M^n, \tag{53}$$

$$-2d_3 p\theta I_{M-1}^{n+1} + (1+2d_3 p\theta)I_M^{n+1} = 2d_3 p(1-\theta)I_{M-1}^n + d_{IIII} I_M^n + \ell\sigma E_M^n, \tag{54}$$

$$-2d_4 p\theta R_{M-1}^{n+1} + (1+2d_4 p\theta)R_M^{n+1} = 2d_4(1-\theta)R_{M-1}^n + d_{RRR} R_M^n + \ell\kappa E_M^n + \ell\gamma I_M^n, \tag{55}$$

where

$$d_{SSS} = 1-\ell\mu - 2d_1 p(1-\theta), d_{EEE} = 1-\ell\omega_1 - 2d_2 p(1-\theta),$$
$$d_{III} = 1-\ell\omega_2 - 2d_3 p(1-\theta), d_{RRR} = 1-\ell\mu - 2d_4 p(1-\theta),$$

Thus, from (21)–(24) for and $m = 1,\ldots,M-1$, and (48)–(51)the solution vectors $\mathbf{S}^{n+1},\mathbf{E}^{n+1},\mathbf{I}^{n+1}$ and $\mathbf{R}^{n+1}$ may be obtained using the following parallel algorithm:

Processor 1: Solve $A_1\mathbf{S}^{n+1} = B_1\mathbf{S}^n + \mathbf{q_1}$ for $\mathbf{S}^{n+1}$, (56)

Processor 2: Solve $A_2\mathbf{E}^{n+1} = B_2\mathbf{E}^n + \mathbf{q_2}$ for $\mathbf{E}^{n+1}$, (57)

Processor 3: Solve $A_3\mathbf{I}^{n+1} = B_3\mathbf{I}^n + \mathbf{q}_3$ for $\mathbf{I}^{n+1}$, (58)

Processor 4: Solve $A_4\mathbf{R}^{n+1} = B_4\mathbf{R}^n + \mathbf{q_4}$ for $\mathbf{R}^{n+1}$, (59)

where A_1, A_2, A_3, A_4 are a constant, tridiagonal matrix of order $M+1$ and

$$\begin{aligned}
\mathbf{q_1} &= \left[\ell rA_0^n,\ldots,\ell rA_M^n\right]^T, \\
\mathbf{q_2} &= \left[\ell\beta I_0^n C_0^n,\ldots,\ell\beta I_M^n C_M^n\right]^T, \\
\mathbf{q_3} &= \left[\ell\sigma E_0^n,\ldots,\ell\sigma E_M^n\right]^T, \\
\mathbf{q_4} &= \left[\ell\kappa E_0^n + \ell\gamma I_0^n,\ldots,\ell\kappa E_M^n + \ell\gamma I_M^n\right]^T,
\end{aligned} \tag{60}$$

T denoting transpose, is a constant vector of order $M+1$. The square matrices $B_1 = B_1(\mathbf{S}^n), B_2 = B_2(\mathbf{E}^n)$,

$B_3 = B_3(\mathbf{I}^n)$ and $B_4 = B_4(\mathbf{R}^n)$ are also of order $M+1$.

The element of $A_i, B_i, i = 1,\ldots 4$, are easily obtained from (47)–(54) and (21)–(24) with $m = 1, 2, \ldots, M$.

Method: $DM(\theta)$:

Taking $m = 0, M$ in (13)–(16) and using (47) gives $m = 0$,

$$(d_S + \ell\beta B_0^n)S_0^{n+1} - 2d_1 p\theta S_1^{n+1} = \left\{1-2d_1 p(1-\theta)\right\}S_0^n + 2d_1(1-\theta)S_1^n + \ell rA_0^n, \tag{61}$$

$$(d_E - \ell\beta C_0^n)E_0^{n+1} - 2d_2 p\theta E_1^{n+1} = \left\{1-2d_2 p(1-\theta)\right\}E_m^n + 2d_2(1-\theta)E_1^n + \ell\beta I_0^n C_0^n, \tag{62}$$

$$d_I I_0^{n+1} - 2d_3 p\theta I_1^{n+1} = \left\{1-2d_3 p(1-\theta)\right\}I_0^n + 2d_3 p(1-\theta)I_1^n + \ell\sigma E_0^n, \tag{63}$$

$$d_R R_0^{n+1} - 2d_4 p\theta R_1^{n+1} = \left\{1-2d_4 p(1-\theta)\right\}R_0^n + 2d_4(1-\theta)R_1^n + \ell\kappa E_0^n + \ell\gamma I_0^n, \tag{64}$$

and $m = M$,

$$-2d_1 p\theta S_{M-1}^{n+1} + \left(d_S + \ell\beta B_M^n\right)S_M^{n+1} = 2d_1(1-\theta)S_{M-1}^n + \left\{1-2d_1 p(1-\theta)\right\}S_M^n + \ell rA_M^n, \tag{65}$$

$$-2d_2 p\theta E_{M-1}^{n+1} + \left(d_E - \ell\beta C_M^n\right)E_M^{n+1} = 2d_2(1-\theta)E_{M-1}^n + \left\{1-2d_2 p(1-\theta)\right\}E_M^n + \ell\beta I_M^n C_M^n, \tag{66}$$

$$-2d_3 p\theta I_{M-1}^{n+1} + d_I I_M^{n+1} = 2d_3 p(1-\theta)I_{M-1}^n + \left\{1-2d_3 p(1-\theta)\right\}I_M^n + \ell\sigma E_M^n, \tag{67}$$

$$-2d_4 p\theta R_{M-1}^{n+1} + d_R R_M^{n+1} = 2d_4(1-\theta)R_{M-1}^n + \left\{1-2d_4 p(1-\theta)\right\}R_M^n + \ell\kappa E_M^n + \ell\gamma I_M^n, \tag{68}$$

In this method the solution vectors $\mathbf{S}^{n+1},\mathbf{E}^{n+1},\mathbf{I}^{n+1}$and $\mathbf{R}^{n+1}$ may be obtained using the parallel algorithm:

Processor 1: Solve $C_1\mathbf{S}^{n+1} = D_1\mathbf{S}^n + \mathbf{q_1}$ for $\mathbf{S}^{n+1}$, (69)

Processor 2: Solve $C_2\mathbf{E}^{n+1} = D_2\mathbf{E}^n + \mathbf{q_2}$ for $\mathbf{E}^{n+1}$, (70)

Processor 3: Solve $C_3\mathbf{I}^{n+1} = D_3\mathbf{I}^n + \mathbf{q}_3$ for $\mathbf{I}^{n+1}$, (71)

Processor 4: Solve $C_4\mathbf{R}^{n+1} = D_4\mathbf{R}^n + \mathbf{q_4}$ for $\mathbf{R}^{n+1}$, (72)

in which the matrices $C_i, D_i, i = 1,\ldots 4$, are square and of order $M+1$. The element of $D_1 = D_1(\mathbf{S}^n), D_2 = D_2(\mathbf{E}^n)$,

$D_3 = D_3(\mathbf{I}^n)$ and $D_4 = D_4(\mathbf{R}^n)$, and the constant matrices $C_i, i = 1,\ldots 4$, may be obtained from (61)–(68), (13)–(16) with $m = 1, 2, \ldots, M$.

As described above, the linear algebraic systems given by (56)–(59) and (69)–(72) can be solved using parallel computation (using a computer with two processors). In parallel computation, the vectors $\mathbf{S}^{n+1},\mathbf{E}^{n+1},\mathbf{I}^{n+1}$ and $\mathbf{R}^{n+1}$ can be obtained simultaneously and thus the time taken to solve the initial-boundary-value problem (1)–(6) will be reduced significant.

V. Numerical Experiments

A. Experiment I: Effect of Time-Step, ℓ

In order to test the stability and convergence properties of the novel scheme constructed in Section II, the model (1)–(6) is simulated by using $DM(\theta)$ and $SM(\theta)$ with the parameter values: $\gamma = 0.20$ year^{-1}, $\mu = 5.5\times10^{-5}$ day^{-1}, $r = 0.0714$day^{-1}, $K = 1.0$, $\sigma = 0.50$ day^{-1}, κ=0.1857 day^{-1}, α=0.0093 day^{-1} and β=0.514 day^{-1}. Furthermore, the following initial subpopulations are for simulation purposes: for $-2 \le x \le 2$,

$$S(x,0) = 0.96\exp\left(-10x^2\right),\ E(x,0) = 0,$$
$$I(x,0) = 0.04\exp(-100x^2),\ R(x,0) = 0,$$

as shown in Fig. I. It shows initial proportion of susceptible and infected individuals concentrated at the origin in which the proportion of susceptible individual is greater than the proportion of the infected individual.

The effect of the time-step on the methods $SM(\theta)$ and $DM(\theta)$ with $\theta = 0, \frac{1}{2}, 1$ is monitored for the different values of the diffusion rate d_1 and fixed the other diffusion rate: $d_i, i = 2,3,4.$ The numerical results are obtained using $d_1 = 0.001,\ 0.01,\ 0.1$ and th**e** stability intervals of the numerical methods are shown in Table I. In the case method $SM(\theta)$ with $\theta = 0$ and $d_1 = 0.001,$ this method produces overflow when $\ell > 0.1896$ while contrived oscillations were exhibited in the numerical solutions as ℓ is increased beyond the value 0.1899 with $d_1 = 0.01$ and the value 0.0500 with $d_1 = 0.1.$ Using $\theta = \frac{1}{2}$, method $SM(\theta)$ produces overflow as $\ell > 2.5533$ with $d_1 = 0.001,$ $\ell > 2.9240$ with $d_1 = 0.01$ and $\ell > 2.6921$ with $d_1 = 0.1.$

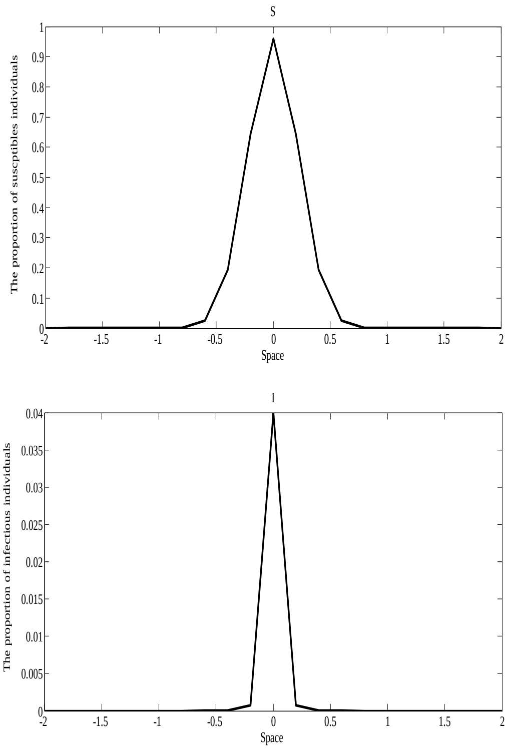


Fig. 1. The initial distributions of the proportion of susceptibles, exposed, infectious and recovered individuals for Experiments I.

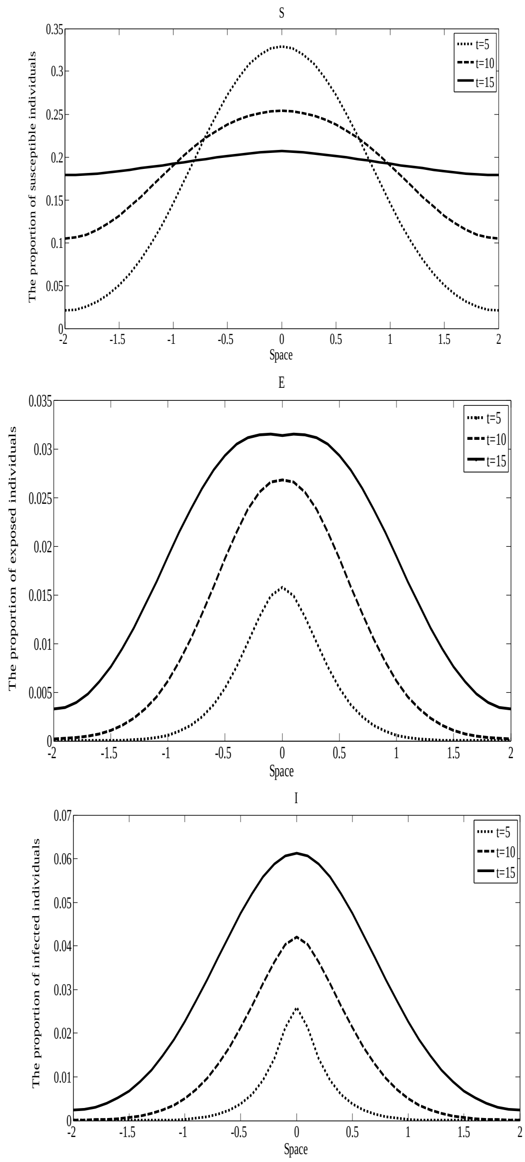
TABLE I: STABILITY INTERVALS OF THE METHODS

d_1	θ	Interval of stability	
		Method $SM(\theta)$	Method $DM(\theta)$
0.001	0	(0, 0.1896)	(0, 0.2136)
	$\frac{1}{2}$	(0, 2.5533)	(0, 11.8433)
	1	(0, 2.7999)	(0, 18.3880)
0.01	0	(0, 0.1899)	(0, 0.2136)
	$\frac{1}{2}$	(0, 2.9240)	(0, 11.0000)
	1	(0, 2.9243)	(0, 14.8312)
0.1	0	(0, 0.0500)	(0, 0.0500)
	$\frac{1}{2}$	(0, 2.6921)	(0, 16.4129)
	1	(0, 2.9238)	(0, 17.6499)

Using $\theta = 1,$ method $DM(\theta)$ produces overflow for $\ell > 2.7999$ with $d_1 = 0.001,$ $\ell > 2.9243$ with $d_1 = 0.01$ and $\ell > 2.9238$ with $d_1 = 0.1.$

For method $DM(\theta)$ with $\theta = 0,$ this method produces overflow for $\ell > 0.2136$ with $d_1 = 0.001, 0.01$ and for $\ell > 0.0500$ with $d_1 = 0.1.$ Using $\theta = \frac{1}{2}$ and $\theta = 1$ method $DM(\theta)$ produces overflow began as ℓ is increased above the values in the stability interval (see Table I).

As above described and Table I, it is verified that method $DM(\theta)$ has a much better stability property than method $SM(\theta).$ Furthermore, the method $DM(\theta)$ with $\theta = 1$ will be used to simulate the model (1)–(6). The space and time steps are given the value $h = 0.1$ and ℓ=0.01, respectively. The numerical results at $t = 5,10,15$ are depicted in Fig. 2. These show that the proportion of all populations spread throughout domain $[-2, 2]$ as time increases. The obtained results appear to be almost coincident with Samsuzzoha *et al*. [1] in their Fig. 2, therefore, showing that the proposed method (14)-(17) gives a reliable representation of the numerical solutions associated with (1)-(7).



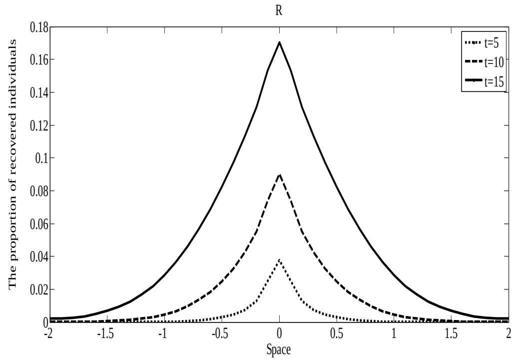


Fig. 2. Numerical simulation of the model at time t=5,10,15 with $h = 0.1$, ℓ=0.01: a) Proportion of susceptible individual on $-2 \le x \le 2$; b) Proportion of exposed individual on $-2 \le x \le 2$; c) Proportion of infectious individual on $-2 \le x \le 2$; d) Proportion of recovered individual on $-2 \le x \le 2$.

VI. Conclusions

A first-order in time and second-order in space, finite-difference scheme has been developed and implemented in this paper for computing the solutions of the *SEIR* influenza model in one dimension (1)–(4). The von Neumann method is used to investigate the stability of the scheme. Numerical experiments are chosen to investigate the dynamic behavior of the model for different step-lengths. It is seen that the proposed method is more competitive (in terms of numerical stability) than the standard finite-difference method. Our results also show that the proposed method (the method $DM(\theta)$) produced numerically-stable solutions for the *SEIR* influenza model which are similar to those reported in Samsuzzoha *et al*. [1].

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