The Equivalence of Commutativity and Independence

Ilija Barukčić

***Abstract*—The concept of mutual independence is related to basic topics of science as such and especially to relativity and quantum theory. Under which circumstances can something be treated as being free from any influence, guidance or control of another? In fact, can something be 'absolutely' independent of another or of itself at all? Is the concept of independence reference-frame dependent? While the probability based concept of independence is solved in a logically consistent way, the relationship between independence and commutativity is still a matter of dispute. This publication will make the proof that commutativity and independence are equivalent.**

***Index Terms*—Independence, dependence, commutativity, non-commutativity, causality.**

I. Introduction

The concept of independence addresses the central problems of philosophy, physics and especially of the theory of probability [1]. Under which circumstances can we "make precise the premises which would make it possible to regard any given real events as independent" [2]? Historically, the traditional, probability based approach to the concept of mutual independence is backgrounded by De Moivre's position. "Two Events are independent, when they have no connexion one with the other, and that the happening of one neither forwards nor obstructs the happening of the other. Two events are dependent, when they are so connected together as that the Probability of either's happening is altered by the happening of the other." [3]. In last consequence, according to De Moivre "those two Events being independent, the Probability of their both happening will be 1/13 × 1/13 = 1/169" [4]. In general, if we follow Kolmogorov's more axiomatic definition of independence [5], we must accept that

$$p\left(Y_t \cap X_t\right) = p\left(Y_t\right) \times p\left(X_t\right) \tag{1}$$

where $p\ (Y_t \cap X_t)$ denotes the joint probability function of $Y_t \cap X_t$, $p\ (Y_t)$ is the probability function of Y_t and $p\ (X_t)$ is the probability function of X_t. In general, under condition of independence, the joint probability $p\ (Y_t \cap X_t)$ equals the product of the single probabilities $p\ (Y_t)$ and $p\ (X_t)$.

II. Material and Methods

A. Commutator

In the paper "Essai sur un nouveau mode d'exposition des principes du calcul differential" [6] published 1814 by François Joseph Servois (1768-1847) in *Annales des mathématiques pures et appliquées* (often called Annales de Gergonne), the words commutative and distributive were used for the first time in history in their mathematical sense. In general, changing the order of operands can but must not change the result.

Thus far, let Y denote something, an expectation value, a quantum mechanical operator, a (suitable) tensor et cetera, let X denote something other, another expectation value, another quantum mechanical operator, another (suitable) tensor et cetera. Under conditions of commutativity it is

$$Y \times X = X \times Y \tag{2}$$

In general, the commutator $[Y, X]$ is defined as

$$\left[Y, X\right] = Y \times X - X \times Y \tag{3}$$

In today's physics, the commutator $[Y, X]$ of two operators Y and X acting on a Hilbert space is a fundamental concept in all areas of quantum physics. The commutator $[Y, X]$ of two operators Y and X is zero if and only if Y and X commute, otherwise not. In other words, if the commutator $[Y, X]$ of two operators Y and X is equal to zero or if

$$\left[Y, X\right] = Y \times X - X \times Y = 0 \tag{4}$$

then the two operators Y and X do commute, otherwise not. Non-commutativity is thus far the absence of commutativity, the other or the complementary of commutativity.

The following Fig. 1 illustrates non-commutativity. Let N and O denote some operands. Changing the order of these operands can yield a different result.

NO≠ON

Fig. 1. Non-commutativity.

B. Anti-Commutator

In general, the anti-commutator $\{Y, X\}$ of Y and X is defined as

$$\left\{Y, X\right\} = Y \times X + X \times Y \tag{5}$$

where Y denotes something, an expectation value, a quantum mechanical operator, a (suitable) tensor et cetera and X denotes something else, another expectation value, another quantum mechanical operator, another (suitable) tensor et cetera.

C. The Product of $Y \times X$

The product $Y \times X$ follows as

Manuscript received May 25, 2013; revised December 23, 2013.

Ilija Barukčić is with Horandstrasse, Jever, Germany (e-mail: Barukcic@t-online.de).

DOI: 10.7763/IJAPM.2013.V3.238

$$Y \times X = \frac{Y \times X - X \times Y + Y \times X + X \times Y}{2} = \frac{[Y, X] + \{Y, X\}}{2} \tag{6}$$

where *Y* denotes something and *X* denotes something else.

D. The Commutative Multiplication $Y \cap X$

Under secured conditions of commutatitiy of *Y* and *X* (4), the commutative multiplication of $Y \cap X$ follows as

$$\begin{aligned} Y \cap X &= Y \times X \\ &= \frac{[Y, X] + \{Y, X\}}{2} \\ &= \frac{Y \times X - X \times Y + Y \times X + X \times Y}{2} \\ &= \frac{0 + Y \times X + X \times Y}{2} \\ &= \frac{\{Y, X\}}{2} \end{aligned} \tag{7}$$

where *Y* denotes something and *X* denotes something else.

III. Results

There are many ways to discuss the relation between dependence and independence [7] while the interior logic between independence and commutativity is still not worked out in an appropriate way. Why should (the occurrence of) something affect (the occurrence of) something other at all? Under which circumstances can we be sure, that (the occurrence of) the one does not affect (the occurrence of) another (i.e. its own other) at all? In general, under which conditions and circumstances are we logically "authorized" to regard something (observed or measured) existing independently of the human mind and consciousness as being independent of something other (observed or measured) existing independently of the human mind and consciousness too?

A. The Equivalence of Commutativity and Independence

In what follows, let *Y* denote something, let *X* denote something else, let [*Y*, *X*] denote the commutator of *Y* and *X*. Further, let

$$\frac{Y}{Y} = \left(\frac{1}{Y}\right) \times \frac{Y}{1} = \left(\frac{1}{Y}\right) \times Y = Y \times \left(\frac{1}{Y}\right) = 1 \tag{8}$$

Claim.

Under these assumptions (8), *Y* and *X* are independent only if the commutator [*Y*, *X*] is equal to zero, otherwise not. In general, the independence of *Y* and *X* is determined by

$$[Y, X] = Y \times X - X \times Y = 0 \tag{9}$$

Proof.

In general, it is

$$+1 = +1. \tag{10}$$

Multiplying by *X*, we obtain

$$+1 \times X = +1 \times X \tag{11}$$

which is equivalent to

$$+X = +X \tag{12}$$

or to

$$+1 \times X = +X \times 1 \tag{13}$$

Rearranging (13) yields

$$+1 \times X - X \times 1 = 0 \tag{14}$$

According to (8) we obtain

$$+\frac{Y}{Y} \times X - X \times \frac{Y}{Y} = 0 \tag{15}$$

Equation (8) demands that (*Y*/*Y*) =1. Under these circumstances *Y* can take any possible value, *X* will stay that what it is, *X* will not change at all. *Y* can change in any direction; under these circumstances (8), *Y* will have no influence on *X*. As long as (15) is valid, any change of *Y* has no influence on the change *X* and vice versa. Under conditions of (15) we can we be sure, that (the change of) the one does not affect (the change of) another (i.e. its own other) at all. Under conditions of (15) *Y* and *X* must be regarded as being independent. From (15) follows that

$$+\frac{1}{Y} \times Y \times X - X \times Y \times \frac{1}{Y} = 0 \tag{16}$$

(according to (8)) or that

$$+\frac{1}{Y}(Y \times X - X \times Y) = 0 \tag{17}$$

which is equivalent to

$$+\frac{1}{Y} \times \frac{Y \times X - X \times Y}{1} = 0 \tag{18}$$

or to

$$+\frac{Y \times X - X \times Y}{Y} = 0 \tag{19}$$

Equation (19) is equivalent to

$$+\frac{Y \times X - X \times Y}{Y} = +\frac{[Y, X]}{Y} = 0 \tag{20}$$

Multiplying (20) by *Y* yields

$$+[Y, X] = 0. \tag{21}$$

Q. e. d.

In the case of independence of Y and X, the commutator of Y and X is equal to zero, otherwise not. Non-commutativity as the other of commutativity is equally the other, the absence of independence [8].

While measuring the degree of non-commutativity we are able to measure the degree of dependence. Non-commutativity may be deeply related to causation but cannot be regarded as being identical with causation. Note well that causation and non-commutativity are not identical but different.

IV. Discussion

Finally, based on our proof above, we must confirm the equivalence of independence and commutativity.

Based on (3), (4), (5), (6) and (7), in the case of independence it is equally

$$+\frac{\{Y,X\}}{2}=\frac{+[Y,X]+\{Y,X\}}{2}. \tag{22}$$

or

$$+Y\cap X=Y\times X \tag{23}$$

In general, in the case of independence we obtain

$$+Y\cap X-Y\times X=0 \tag{24}$$

The equivalence of commutativity and independence is proofed as valid under circumstances where $Y/Y = 1$. But does this circumstance exist at all?

Under condition of (8) Y and X can be treated as (self-adjoint) operators (on a Hilbert space) or as quantum mechanical observables too. Quantum mechanical operators do not necessarily commute with each other. As soon as the commutator of Y and X is not equal to zero, we can be sure that Y and X can no longer be treated as being independent. What happens if you divide a quantum mechanical (self-adjoint) operator Y by itself?

Algebraic operations for tensors like addition and subtraction are valid between tensors of the same rank and are both commutative and associative. The multiplication of a tensor of the same rank by itself or another tensor of the same rank is not commutative per se.

Today, a division of a tensor of the same rank by itself or another tensor of the same rank is not possible in a logically consistent way. But our proof above is unrestrictedly valid for tensors too. Therefore, it seem justified to claim that the proof above is useless for tensor calculus as long as the tensor calculus is not enriched or extended by the possibility of the division of a (suitable) tensor by another (suitable) tensor.

Let us once again stress out that our proof above is based on the assumption that $Y/Y = Y \times (1/Y) \times (1/Y) \times Y = 1$. This allows too that $Y = 0$. But what happens if we divide by 0? Is $(0/0) = 0 \times (1/0) = 1$? It is important to note, that our proof does not exclude a division by zero but equally the same proof does not make any evidence that a division of zero by zero is allowed, possible or logically consistent. The problem of the division of zero by zero is not part of this publication.

The case $Y = +\infty$ or $Y = -\infty$ is not excluded from our proof too. As long as it is true that $+\infty/+\infty = +\infty \times (1 / +\infty) = 1$, our proof above is valid under these circumstances too. In striking contrast to expectation, our proof above has not provided any evidence that $+\infty/+\infty = +\infty \times (1 / +\infty) = 1$, which is important to bear in mind.

In general, under conditions where $Y/Y = Y \times (1/Y) = 1$, we are allowed to accept the equivalence of commutativity and independence. But it is equally another question, under which circumstances this condition is given.

V. Conclusion

Our proof above is important for quantum theory and equally for relativity theory too. The power of this publication lies in the potential to establish a new connection between general relativity and quantum mechanics and to simplify our attempt to unify both theories.

The equivalence of commutativity and independence is proofed as valid. In general, it is possible to measure the degree of non-commutativity very precisely in terms of quantum mechanical operators or even while using the tensor calculus.

References

[1] A. N. Kolmogorov, *Foundations of the Theory of Probability*, Second English Edition, Translation Edited by Nathan Morrison, New York: Chelsea Publishing Company, 1956, pp. 8.

[2] A. N. Kolmogorov, *Foundations of the Theory of Probability*, Second English Edition, Translation Edited by Nathan Morrison, New York: Chelsea Publishing Company, 1956, pp. 9.

[3] A. de Moivre, *The Doctrine of Chances or A Method of Calculating the Probabilities of Events in Play*, Third Edition, London: A. Millar, 1756, [MDCCLVI], pp. 6.

[4] A. de Moivre, *The Doctrine of Chances or A Method of Calculating the Probabilities of Events in Play*, Third Edition, London: A. Millar, 1756, [MDCCLVI], pp. 7.

[5] A. N. Kolmogorov, *Foundations of the Theory of Probability*, Second English Edition, Translation Edited by Nathan Morrison, New York: Chelsea Publishing Company, 1956, pp. 9.

[6] F. J. Servois, "Essai sur un nouveau mode d'exposition des principes du calcul différentiel," *Annales de mathématiques pures et appliquées*, vol. 5, pp. 93-140, 1814.

[7] A. N. Kolmogorov, *Foundations of the Theory of Probability*, Second English Edition, Translation Edited by Nathan Morrison, New York: Chelsea Publishing Company, 1956, pp. 60.

[8] B. de Facio and D. C. Taylor, "Commutativity and Causal Independence," *Physical Review D*, vol. 8, issue 8, pp. 2729-2731, 1973.

Ilija Barukčić was born on October 1th, 1961 in Novo Selo (Bosina and Hercegovina, former Yugoslavia). He studies at the University of Hamburg from 1982 to 1989, Germany. He ends of studies with the degree: State exam. His basic field of research is the relationship between cause and effect under conditions of quantum and relativity theory, in biomedical sciences, in philosophy et cetera. Thus far, among my main publications in physics, published by the American Institute of Physics, are the refutation of Heisenberg's uncertainty principle (Ilija Barukčić, "Anti Heisenberg - Refutation Of Heisenberg's Uncertainty Relation," *American Institute of Physics - Conference Proceedings*, Volume 1327, pp. 322-325, 2011), the refutation of Bell's theorem and the CHSH inequality (Ilija Barukčić, "Anti-Bell - Refutation of Bell's theorem," *American Institute of Physics - Conference Proceedings*, Volume 1508, pp. 354-358, 2012.). In another publication, I was able to provide the proof that time and the gravitational field is equivalent (Ilija Barukčić, "The Equivalence of Time and Gravitational Field," *Physics Procedia*, Volume 22, pp. 56-62, 2011).