Some Remarks on the Properties of Double Laplace Transforms

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***Abstract*—In this note, we have discussed and proved the different properties of double Laplace transforms like linearity property, change of scale, shifting property, double Laplace transform of partial derivatives, double Laplace transform of integral, multiplication by *xt* and division by *xt*.**

***Index Terms*—Double laplace transforms, partial derivatives.**

I. Introduction

Partial differential equations have big importance in Mathematics and other fields of Science. Therefore it is very important to know methods to solve such partial differential equations. One of the most known methods to solve partial differential equations is the integral transform method [1], [2]. Eltayeb and Kilicman [3], [4] have established and studied the relationship between the double Sumudu transform and the double Laplace transform and their applications to differential equations. Singh and Mandia [5] have established the relation between the double Laplace transform and the double Mellin transform and discussed their applications. Recently, Eltayeb and Kilicman [6] have applied the double Laplace transform to solve general linear telegraph and partial integro-differential equations.

In this paper, we have discussed the various properties of double Laplace transforms. The Double Laplace Transform is very useful in the solution of many Partial differential equations & it can be used as a very effective tool in simplifying the calculations in many fields of Engineering & Mathematics. We have discussed and proved the different properties of double Laplace transforms like linearity property, change of scale, shifting property, double Laplace transform of partial derivatives, double Laplace transform of integral, multiplication by xy and division by xy.

Let $f(x, t)$ be a function of two variables x and t, where $x, t > 0$. The double Laplace transform of $f(x, t)$ as defined by Kilicman *et al.* [7], [8] is given as

$$L_t L_x\{f(x,t)\} = \overline{f}(s_1, s_2) = \int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x} f(x,t)\,dx\,dt \tag{1}$$

whenever the improper integral converges.

Manuscript received March 12, 2013; revised May 16, 2013.

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II. Properties of Double Laplace Transforms

A. Linearity Property

If $f(x, t)$ and $g(x, t)$ be two functions of x and t such that $L_t L_x\{f(x,t)\} = \overline{f}(s_1, s_2)$ and $L_t L_x\{g(x,t)\} = \overline{g}(s_1, s_2)$ then

$$L_t L_x\{\alpha f(x,t) + \beta g(x,t)\} = \alpha L_t L_x\{f(x,t)\} + \beta L_t L_x\{g(x,t)\}$$

where α and β are constants.

This property follows easily from (1).

B. Change of Scale Property

If $L_t L_x\{f(x,t)\} = \overline{f}(s_1, s_2)$ then

$$L_t L_x\{f(ax, bt)\} = \frac{1}{ab}\overline{f}\left(\frac{s_1}{a}, \frac{s_2}{b}\right)$$

where a and b are constants.

Proof: from (1), we have

$$L_t L_x\{f(ax, bt)\} = \int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x} f(ax, bt)\,dx\,dt \tag{2}$$

We put $ax = u$ and $bt = v$ in the integral of (2), where u and v takes the limit from 0 to ∞. Hence, we get

$$L_t L_x\{f(ax, bt)\} = \int_0^\infty e^{-s_2\left(\frac{v}{b}\right)}\int_0^\infty e^{-s_1\left(\frac{u}{a}\right)} f(u,v)\frac{du}{a}\frac{dt}{b}$$
$$= \frac{1}{ab}\int_0^\infty e^{-s_2\left(\frac{v}{b}\right)}\int_0^\infty e^{-s_1\left(\frac{u}{a}\right)} f(u,v)\,du\,dt = \frac{1}{ab}\overline{f}\left(\frac{s_1}{a}, \frac{s_2}{b}\right)$$

Thus $L_t L_x\{f(ax, bt)\} = \dfrac{1}{ab}\overline{f}\left(\dfrac{s_1}{a}, \dfrac{s_2}{b}\right)$

C. First Shifting Property

If $L_t L_x\{f(x,t)\} = \overline{f}(s_1, s_2)$ then

$$L_t L_x\{e^{ax+bt} f(x,t)\} = \overline{f}(s_1 - a, s_2 - b)$$

where a and b are constants.

Proof: From (1), we have

$$L_t L_x\{e^{ax+bt} f(x,t)\} = \int_0^\infty e^{-s_2 t}\int_0^\infty e^{-s_1 x} e^{ax+bt} f(x,t)\,dx\,dt$$

DOI: 10.7763/IJAPM.2013.V3.224

$$= \int_0^\infty e^{-(s_2-b)t} \int_0^\infty e^{-(s_1-a)x} f(x,t)\,dx\,dt$$

$$= \overline{f}\,(s_1 - a,\ s_2 - b)$$

Thus

$$L_t\, L_x \left\{ e^{ax+bt} f(x,t) \right\} = \overline{f}\ (s_1 - a,\ s_2 - b)$$

D. Double Laplace Transform of Partial Derivatives

If $L_t\, L_x \{ f(x,t) \} = \overline{f}\ (s_1, s_2)$ then

$$L_t\, L_x \left\{ \frac{\partial^2}{\partial x\, \partial t} f(x,t) \right\} = L_t L_x \{ f_{xt}(x,t) \}$$

$$= s_1 s_2 \overline{f}\,(s_1, s_2) - s_1 \overline{f}\,(s_1, 0) - s_2 \overline{f}\,(0, s_2) + f(0,0)$$

Proof: from (1), we have

$$L_t\, L_x \left\{ \frac{\partial^2}{\partial x\, \partial t} f(x,t) \right\} = L_t L_x \{ f_{xt}(x,t) \}$$

$$= \int_0^\infty e^{-s_2 t} \int_0^\infty e^{-s_1 x} f_{xt}(x,t)\,dx\,dt$$

$$= \int_{x=0}^{\infty} e^{-s_1 x} \left\{ \int_{t=0}^{\infty} e^{-s_2 t} f_{xt}(x,t)\,dt \right\} dx$$

$$= \int_{x=0}^{\infty} e^{-s_1 x} \left\{ [e^{-s_2 t} f_x(x,t)]_{t=0}^{\infty} - \int_{t=0}^{\infty} e^{-s_2 t}(-s_2)\, f_x(x,t)\,dt \right\} dx$$

$$= \int_{x=0}^{\infty} e^{-s_1 x} \left\{ 0 - f_x(x,0) + s_2 \int_{t=0}^{\infty} e^{-s_2 t} f_x(x,t)\,dt \right\} dx$$

$$= -\int_{x=0}^{\infty} e^{-s_1 x} f_x(x,0)\,dx + s_2 \int_{x=0}^{\infty} e^{-s_1 x} \int_{t=0}^{\infty} e^{-s_2 t} f_x(x,t)\,dt\,dx$$

$$= -\left\{ [e^{-s_1 x} f(x,0)]_{x=0}^{\infty} - \int_{x=0}^{\infty} e^{-s_1 x}(-s_1) f(x,0)\,dx \right\}$$

$$+ s_2 \int_{t=0}^{\infty} e^{-s_2 t} \left\{ \int_{x=0}^{\infty} e^{-s_1 x} f_x(x,t)\,dx \right\} dt$$

$$= -\left\{ 0 - f(0,0) + s_1 \int_{x=0}^{\infty} e^{-s_1 x} f(x,0)\,dx \right\}$$

$$+ s_2 \int_{t=0}^{\infty} e^{-s_2 t} \left\{ [e^{-s_1 x} f(x,t)]_{x=0}^{\infty} - \int_{x=0}^{\infty} e^{-s_1 x}(-s_1) f(x,t)\,dx \right\} dt$$

$$= f(0,0) - s_1 \overline{f}\,(s_1, 0) + s_2 \int_{t=0}^{\infty} e^{-s_2 t} \left\{ 0 - f(0,t) + s_1 \int_{x=0}^{\infty} e^{-s_1 x} f(x,t)\,dx \right\} dt$$

$$= f(0,0) - s_1 \overline{f}\,(s_1, 0) - s_2 \int_{t=0}^{\infty} e^{-s_2 t} f(0,t)\,dt$$

$$+ s_1 s_2 \int_{t=0}^{\infty} e^{-s_2 t} \int_{x=0}^{\infty} e^{-s_1 x} f(x,t)\,dx\,dt$$

$$= f(0,0) - s_1 \overline{f}\,(s_1, 0) - s_2 \overline{f}\,(0, s_2) + s_1 s_2 \overline{f}\,(s_1, s_2)$$

Thus

$$L_t L_x \{ f_{xt}(x,t) \} = s_1 s_2 \overline{f}\,(s_1, s_2) - s_1 \overline{f}\,(s_1, 0) - s_2 \overline{f}\,(0, s_2) + f(0,0)$$

E. Double Laplace Transform of Integral

If $L_t\, L_x \{ f(x,t) \} = \overline{f}\ (s_1, s_2)$ then

$$L_t\, L_x \left\{ \int_0^x \int_0^t f(u,v)\,du\,dv \right\} = \frac{\overline{f}\ (s_1, s_2)}{s_1 s_2}, \qquad s_1 > 0,\ s_2 > 0.$$

Proof: Let

$$g(x,t) = \int_0^x \int_0^t f(u,v)\,du\,dv$$

Hence we have,

$$g_{xt}(x,t) = f(x,t) \quad and \quad g(0,0) = 0$$

$$L_t L_x \{ g_{xt}(x,t) \} = L_t L_x \{ f(x,t) \} = \overline{f}\,(s_1, s_2)$$

From the property 2.4, we have

$$L_t\, L_x \left\{ \frac{\partial^2}{\partial x\, \partial t} g(x,t) \right\} = L_t L_x \{ g_{xt}(x,t) \}$$

$$= s_1 s_2 \overline{g}\,(s_1, s_2) - s_1 \overline{g}\,(s_1, 0) - s_2 \overline{g}\,(0, s_2) + g(0,0)$$

$$\Rightarrow \overline{f}\,(s_1, s_2) = s_1 s_2 \overline{g}\,(s_1, s_2) - s_1 \overline{g}\,(s_1, 0) - s_2 \overline{g}\,(0, s_2) \triangleleft$$

$$\Rightarrow \overline{g}\,(s_1, s_2) = \frac{1}{s_1 s_2} \overline{f}\,(s_1, s_2) + \frac{1}{s_2} \overline{g}\,(s_1, 0) + \frac{1}{s_1} \overline{g}\,(0, s_2)$$

$$\Rightarrow \overline{g}\,(s_1, s_2) = \frac{1}{s_1 s_2} \overline{f}\,(s_1, s_2) + \frac{1}{s_2} L\{g(x,0)\} + \frac{1}{s_1} L\{g(0,t)\}$$

But $L\{g(x,0)\} = 0$ and $L\{g(0,t)\} = 0$

Therefore

$$\overline{g}\,(s_1, s_2) = \frac{1}{s_1 s_2} \overline{f}\,(s_1, s_2)$$

$$\Rightarrow \quad L_t L_x \{ g(x,t) \} = \frac{1}{s_1 s_2} \overline{f}\,(s_1, s_2)$$

Hence $L_t L_x \left\{ \int_0^x \int_0^t f(u,v)\,du\,dv \right\} = \dfrac{\overline{f}\,(s_1, s_2)}{s_1 s_2}$

F. Multiplication by xt

If $L_t\, L_x \{ f(x,t) \} = \overline{f}\ (s_1, s_2)$ then

$$L_t\, L_x \{ xt\, f(x,t) \} = (-1)^{1+1} \frac{\partial^2}{\partial s_1 \partial s_2} \overline{f}\ (s_1, s_2)$$

Proof: Differentiating (1) partially with respect to s_1 and s_2, we get

$$\frac{\partial^2}{\partial s_1 \partial s_2} \overline{f}\ (s_1, s_2) = (-1)(-1) \int_0^\infty e^{-s_2 t} \int_0^\infty e^{-s_1 x}\, xt\, f(x,t)\,dx\,dt$$

$$\Rightarrow \quad \frac{\partial^2}{\partial s_1 \partial s_2}\overline{f}\,(s_1, s_2) = (-1)^{1+1} L_t L_x \{xt\, f(x, t)\}$$

Thus

$$L_t\, L_x \{xt\, f(x, t)\} = (-1)^{1+1}\frac{\partial^2}{\partial s_1 \partial s_2}\overline{f}\,(s_1,\, s_2)$$

In general,

$$L_t\, L_x \{x^m t^n\, f(x, t)\} = (-1)^{m+n}\frac{\partial^{m+n}}{\partial s_1^m \partial s_2^n}\overline{f}\,(s_1,\, s_2)$$

G. Division by xt

If $L_t\, L_x \{f(x, t)\} = \overline{f}\,(s_1, s_2)$ then

$$L_t\, L_x\left\{\frac{f(x, t)}{xt}\right\} = \int_{s_1}^{\infty}\int_{s_2}^{\infty}\overline{f}\,(s_1,\, s_2)\, ds_1 ds_2$$

Proof: We assume that $\int_{s_1}^{\infty}\int_{s_2}^{\infty}\overline{f}\,(s_1,\, s_2)\, ds_1 ds_2$ exists.

Integrating (1) with respect to s_1 from s_1 to ∞ and s_2 from s_2 to ∞, we get

$$\int_{s_1}^{\infty}\int_{s_2}^{\infty}\overline{f}\,(s_1, s_2)\, ds_1 ds_2 = \int_{s_1}^{\infty}\int_{s_2}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} e^{-s_1 x} e^{-s_2 t}\, f(x, t)\, dx\, dt\, ds_1 ds_2$$

$$\int_{s_1}^{\infty}\int_{s_2}^{\infty}\overline{f}\,(s_1, s_2)\, ds_1 ds_2 = \int_{s_2}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\left[\frac{e^{-s_1 x}}{-x}\right]_{s_1 = s_1}^{\infty} e^{-s_2 t}\, f(x, t)\, dx\, dt\, ds_2$$

$$\int_{s_1}^{\infty}\int_{s_2}^{\infty}\overline{f}\,(s_1, s_2)\, ds_1 ds_2 = \int_{0}^{\infty}\int_{0}^{\infty}\left[0 + \frac{e^{-s_1 x}}{-x}\right]\left[\frac{e^{-s_2 t}}{-t}\right]_{s_2 = s_2}^{\infty} f(x, t)\, dx\, dt$$

$$\int_{s_1}^{\infty}\int_{s_2}^{\infty}\overline{f}\,(s_1, s_2)\, ds_1 ds_2 = \int_{0}^{\infty}\int_{0}^{\infty} e^{-s_1 x} e^{-s_2 t}\,\frac{f(x, t)}{xt}\, dx\, dt = L_t L_x\left\{\frac{f(x, t)}{xt}\right\}$$

Thus $$L_t\, L_x\left\{\frac{f(x, t)}{xt}\right\} = \int_{s_1}^{\infty}\int_{s_2}^{\infty}\overline{f}\,(s_1,\, s_2)\, ds_1 ds_2$$

provided the integral on the right side exists.

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