Existence Theorem for Semilinear Impulsive Functional Differential Equations with Nonlocal Conditions

Haydar Akça, Jamal Benbourenane, and Valéry Covachev

Abstract—**The existence, uniqueness and continuous dependence of a mild solution of a Cauchy problem for semilinear impulsive first and second orderfunctional differential-equations with nonlocal conditions in general Banach spaces are studied. Methods of fixed point theorems, of a C_0 semigroup of operators and the Banach contraction theorem are applied.**

Index Terms—**Functional-differential equations,impulsive, mild solution, nonlocal conditions, semilinear.**

I. Introduction

Many evolutionary processes in nature are characterized by the fact that at certain moments of time they experience an arbitrary change of the states. Therefore, the theory of impulsive differential equations is richer than that of the corresponding non-impulsive differential equations and repeats its development (see monographs [1], [2]. The theory of impulsive differential equations is quite new and is one of the attractive branches of differential equations which has extensive realistic mathematical modeling applications in physics, chemistry, engineering, and biological and medical sciences. The nonlocal condition is a generalization of the classical initial condition. A Banach space is a normed linear space that is a complete metric space with respect to the metric derived from its norm [3], [4]. A normed linear space is a metric space with respect to the metric d derived from its norm, $d(x, y) = \|x - y\|$. If $(X, d)$is a metric space, then a contraction of X (also called contraction on X) is a function$f: X \to X$ that satisfies

$$\forall x, x' \in X:\ \ d\big(f(x), f(x')\big) \leq \alpha d(x, x')$$

for some real number $\alpha < 1$. Such an α is called a contraction modulus of f. Every contraction mapping on a complete metric spacehas a unique fixed point. (This is also called the Contraction Mapping Theorem.) If T is a contraction on a complete metric space (X, d) and α is a contraction modulus of$T$, then for every$x \in X$,

$$\forall n \in \mathbb{N}:\ \ d(T^n x, x^*) \leq \alpha^n d(x, x^*),$$

where x^* is the unique fixed point of$T$. The generator A of a C_0semigroup$U$ is the operator

$$Ax \coloneqq \lim_{h\to 0} \frac{1}{h}(U^h - 1)x\,,$$

defined for all those $x \in X$ for which this limit exists. Generally speaking, we say that a C_0semigroup is a strongly continuous one parameter semi-group of bounded linear operators on a Banach space X. AC_0semigroup (or strongly continuous semigroup) is a family $T = \{T(t) | t \in \mathbb{R}^+\}$ of bounded linear operators from X to X satisfying $T(t + s) = T(t)T(s)\ \forall t, s \in \mathbb{R}^+$, secondly, $T(0) = I -$ the identity operator on X and, finally, $\lim_{t\to 0^+} T(t)f = f$ for each $f \in X$ with respect to the norm on X. In this paper we study the existence,uniqueness and continuous dependence of a mild solution of anonlocal Cauchy problem for a semilinear impulsive functional-differential evolution equation. Such problems arise in some physical applications as a natural generalizationof the classical initial value problems. We cannot measure $x(t_0)$ and $x(t_k)$, (k=1,2,…,p)but we can measure a relation between measure $x(t_0)$ and $x(t_k)$, (k=1,2,…,p). There are some physical problems where nonlocal conditions are more realistic and usefulthan the classical initial conditions such as theory of vibrations, kinematicsto determine the evaluation of $t \to x(t)$ of thelocation of a physical object forwhich we do not know the positions $x(t_0), x(t_1), \ldots, x(t_p)$, but we do know thatthe introduced suitable nonlocal condition is satisfied. For examplefollowing nonlocalmodel describes simultaneous description of the motion trajectories $t \to x(t)$ of two rockets. The first rocket is lunched to the cosmic space from the Earthsurface outside of the Earth gravitation and the second rocket lunched fromEarth orbit and next rocket goes to the cosmic space in the same direction as the first rocket [8].

$$\begin{cases} \dot{x}(t) + A(t)x(t) = f\big(t, x(t)\big), \qquad t \in [0, T], \\ x(t) = Me^{\beta t_0} x(t + T^*), t \in [-r, 0], T^* \in [t_0 + r, T] \end{cases}$$

The result shows that for each time$T^* \in [t_0 + r, T]$and delay $\tau \in [-r, 0]$ there exist a unique initial location $\varphi_*(\tau)$ of the rocket which is having unique orbitand there is only one motion trajectory $t \to x_*(t)$; $t \in [-r, T]$ of the rocket. For$t \to \infty$ all the motion trajectories $t \to x_\varphi(t)$of the rocket, corresponding to thelocal problem will tend exponentially to the trajectory $t \to x_*(t)$.

The results for a semilinear functional-differential evolution nonlocal problem [5]-[7] are extended for the case

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of impulse effect.

We consider a nonlocal Cauchy problem in the form

$$\begin{cases} \dot{u}(t) + Au(t) = f\left(t, u(t), u\big(b_1(t)\big), \dots, u\big(b_m(t)\big)\right), \\ \qquad t \in (t_0, t_0 + a], \quad t \neq \tau_k, \\ u(\tau_k + 0) = Q_k u(\tau_k) \equiv u(\tau_k) + I_k u(\tau_k), \quad k = 1,2,\dots,\kappa, \\ u(t_0) = u_0 - g(u), \end{cases} \tag{1}$$

where $t_0 \geq 0$, $a > 0$ and $-A$ is the infinitesimal generator of a compact C_0 semigroup of operators on a Banach space E. $I_k (k = 1,2,\dots,\kappa)$ are linear operators acting in the Banach space E. The functions $f, g, b_i (i = 1, 2, \dots, m)$ are given satisfying some assumptions and u_0 is an element of the Banach space E. $I_k u(\tau_k) = u(\tau_k + 0) - u(\tau_k - 0)$ and the impulsive moments τ_k are such that $t_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots < \tau_\kappa < t_0 + a$, $\kappa \in \mathbb{N}$.

Theorems about the existence, uniqueness and stability ofsolutions of differential and functional-differential abstract evolution Cauchy problems were studied in [8], [9]. The results presented in this paper are a generalization and a continuation of some results reported in[10],[11]. We consider a classical semilinear impulsive functional-differential equation in the case of a nonlocal condition, reduced to the classical impulsive initial functional value problem. So far there have been a limited number of papers in this direction studying the existence of solutions of second order impulsive differential equations with nonlocalconditions. We try to extend the results of first order impulsive differential equations to second order impulsive differential equations with nonlocal conditions.

As usual, in the theory of impulsive differential equations [6], at the points of discontinuity τ_i of the solution $t \mapsto u(t)$ we assume that $u(\tau_i) \equiv u(\tau_i - 0)$. It is clear that, in general, the derivatives $\dot{u}(\tau_i)$ do not exist. On the other hand, according to the firstequality of (1)there exist the limits $\dot{u}(\tau_i \mp 0)$. According to the above convention, we assume $\dot{u}(\tau_i) \equiv \dot{u}(\tau_i - 0)$ [1], [2].

Throughout the paper we assume that E is a Banach space with norm $\|\cdot\|$, $-A$ is the infinitesimal generator of a C_0semigroup$\{T(t)\}_{t\geq 0}$on E, $D(A)$ is the domainof A [11]. A C_0semigroup$\{T(t)\}_{t\geq 0}$is said to be acompact C_0semigroup of operators on E if $T(t)$ is a compact operator for every $t > 0$. We denote

$$I := [t_0, t_0 + a], \qquad M := \sup_{t\in[0,a]} \left\{\|T(t)\|_{BL(E,E)}\right\}$$

and X is the space of piecewisecontinuous functions $I \to E$ with discontinuities of the first kindat $\tau_1, \tau_2, \dots, \tau_\kappa$.

Let $f: I \times E^{m+1} \to E$, $g: X \to E$ (for instance, we can have

$$g(u) = \tilde{g}\left(u(t_1), u(t_2), \dots, u\big(t_p\big)\right),$$

where $\tilde{g}: E^p \to E$, $t_0 < t_1 < t_2 < \cdots < t_p < t_0 + a$, $p \in \mathbb{N}$, $b_i: I \to I (i = 1,2,\dots, m)$ and $u_0 \in E$. In thesequel, the operator norm $\|\cdot\|_{BL(E,E)}$ will be denoted by$\|\cdot\|$. We need the following sets:

$E_\rho := \{z \in E,\ \|z\| \leq \rho\}$ and $X_\rho := \{w \in X,\ \|w\|_X \leq \rho\}$, $\rho > 0$.

Introduce the following assumptions [6]:

A1: $f \in C\ (I \times E^{m+1} \to E)$, $g \in C(X, E)$ and $b_i \in C(I, I)$, $i = 1,2,\dots,m$ and there areconstants C_i, $i = 1,2,3$, such that

$$\begin{cases} \|f(s, z_0, z_1, \dots, z_m)\| \leq C_1 \ for\ s \in I, \\ \qquad z_i \in E_r (i = 0,1,\dots,m), \\ \|g(w)\| \leq C_2 \ and \ \max\limits_{k=1,2,\dots,\kappa} \|I_k w\| \leq C_3 \ for\ w \in X_r, \\ \qquad where\ r := M(aC_1 + \|u_0\| + C_2 + \kappa C_3). \end{cases} \tag{2}$$

A2: $g(\lambda w_1 + (1-\lambda)w_2) = \lambda g(w_1) + (1-\lambda)g(w_2)$ for $w_i \in X_r$, $i = 1,2$, and $\lambda \in (0,1)$, and r is given by (2).

A3: The set $\{w(t_0) = u_0 - g(w): w \in X_r\}$, where r is given by (2), is precompact in E.

Consider the initial value problem [8]

$$\begin{cases} \dot{u}(t) + Au(t) = f(t), \quad t \in (t_0, t_0 + a], \\ u(t_0) = x, \end{cases} \tag{3}$$

where $f: I \to E$, $-A$ is the infinitesimal generator of a C_0semigroup $T(t)$,$t \geq 0$, and $x \in E$.

Definition 1.A function u is said to be a *strong solution* of problem (3) on I if $u$is differentiable almost everywhere (a.e.)on$I$, so that$(du/dt) \in L^1\big((t_0, t_0 + a); E\big)$,$u(t_0) = x$ and$\dot{u}(t) + Au(t) = f(t)$a.e. on I.

The unique strongsolution u on I is given by the formula

$$u(t) = T(t - t_0)x + \int_{t_0}^{t} T(t - s)f(s)\, ds, \quad t \in I. \tag{4}$$

Definition 2.A function u is said to be a *classical solution* of problem (3) on I if u is continuous on I and continuously differentiable on $t \in (t_0, t_0 + a]$, such that $u(t) \in D(A)$for $t_0 < t \leq t_0 + a$ and theproblem (3) is satisfied on$I$.

If E is a Banach space and $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$, $f: I \to E$ is continuous on I and $x \in D(A)$, then the problem (3) has a classical solution u on I given by (4).

Next consider the initial value problem for the impulsive linear system

$$\begin{cases} \dot{u}(t) + Au(t) = f(t), \quad t \in (t_0, t_0 + a], \ t \neq \tau_k, \\ u(\tau_k + 0) = u(\tau_k) + I_k u(\tau_k), \quad k = 1,2,\dots,\kappa, \\ u(t_0) = x, \end{cases} \tag{5}$$

where A, f and x are as in problem (3), and τ_k and I_k are as in problem (1).

Definition 3. A function $u: I \to E$ is said to be a *classicalsolution* of the problem (5) on I if u is piecewise continuous on I with discontinuities of the first kind at $\tau_1, \tau_2, \dots, \tau_\kappa$ and continuouslydifferentiable on $(t_0, t_0 + a \setminus \tau kk{=}1\kappa$, such that *ut∈DA* for *t0<t≤t0+a* and the problem (5) is satisfied on I.

If A, f and x are as above and$I_k: D(A) \to D(A)$, then the Problem(5) has a classical solution u on I given by the formula

$$u(t) = T(t - t_0)x + \int_{t_0}^{t} T(t - s)f(s)\, ds + \sum_{t_0 \leq \tau_k < t} T(t - \tau_k) I_k u(\tau_k). \tag{6}$$

Formula (6) motivates us to give the following definition.

Definition 4.A function $u \in X$ satisfying the following integro-summary equation

$$u(t) = T(t-t_0)u_0 - T(t-t_0)g(u) + \int_{t_0}^{t} T(t-s)f\left(s, u(s), u\left(b_1(s)\right), \dots, u\left(b_m(s)\right)\right) ds + \sum_{t_0 \le \tau_k < t} T(t-\tau_k)I_k u(\tau_k), \quad t \in [t_0, t_0+a],$$

is said to be a *mild solution* of the nonlocal Cauchy problem (1).

II. Existence and UniquenessTheorems

Theorem 1.*Suppose that assumptions***A1-A3** *aresatisfied, then the impulsive nonlocal Cauchy problem(1)has a mild solution.*

Proof. The mild solution of the impulsive system (1) with nonlocal condition satisfies the operator equation (6)

$$u(t) = (Fu)(t),$$

where

$$(Fw)(t) := T(t-t_0)u_0 - T(t-t_0)g(w) + \int_{t_0}^{t} T(t-s)f\left(s, w(s), w\left(b_1(s)\right), \dots, w\left(b_m(s)\right)\right) ds + \sum_{t_0 \le \tau_k < t} T(t-\tau_k)I_k w(\tau_k), \quad t \in [t_0, t_0+a],$$

so that

$$\|(Fw)(t)\| \le M\|u_0\| + MC_2 + aMC_1 + \kappa MC_3 = r,$$

where the impulsive moments τ_k are such that $t_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots < \tau_\kappa < t_0 + a,\ \kappa \in \mathbb{N}$. See detailed proof in [6].

Theorem 2.*Suppose that the functions f, g and $b_i(i=$ 1,2,msatisfyassumptions***A1**, **A2**, *where u0∈E. Then inthe class of all the functions w, for which assumption***A3** *holds, the nonlocal Cauchyproblem (1) has a mild solution u. If in addition:*

1) *E is a reflexive Banach space,*
2) *there exists a constant $L > 0$ such that*

$$\begin{cases} \|f(s, u_0, u_1, \dots, u_m) - f(\tilde{s}, \tilde{u}_0, \tilde{u}_1, \tilde{u}_m)\| \\ \le L_1\left(|s-\tilde{s}| + \sum_{k=0}^{m}\|u_k - \tilde{u}_k\|\right), \\ \|I_k v\|_E \le L_2\|v\|_E \; for \; v \in E, \quad k = 1,2,\dots,\kappa, \end{cases}$$

where $s, \tilde{s} \in I$, $u_i, \tilde{u}_i \in E_r (i = 0,1,2,\dots,m)$ and $L = max \ \{L_1, L_2\}$.

3) *u is the unique mild solution of the problem(1)and there is a constant $K > 0$ such that*

$$\left\|u\left(b_i(s)\right) - u\left(b_i(\tilde{s})\right)\right\| \le K\|u(s) - u(\tilde{s})\| \; for \; s, \tilde{s} \in I,$$

4) *The element$u_0 \in D(A)$and $g(u) \in D(A)$,*

then u is the unique classical solution of the impulsivenonlocalCauchy problem (1).

Proof. Since all the assumptions of Theorem 1 are satisfied,then the nonlocal impulsive Cauchy problem (1) possesses a mild solution u which, according to assumption (*iii*), is the unique mild solution of the problem (1). Now, we will showthat u is the unique classical solution of the semilinear, nonlocal, impulsive, Cauchy problem (1). Therefore,

$$\begin{aligned} u(t+h) - u(t) = {} & [T(t+h-t_0)u_0 - T(t-t_0)u_0] \\ & -[T(t+h-t_0)g(u) - T(t-t_0)g(u)] \\ & + \int_{t_0}^{t_0+h} T\,(t+h-s)f(s, u(s), u(b_1(s)), u(b_2(s)), \dots, u(b_m(s)))\, ds \\ & + \int_{t_0+h}^{t+h} T\,(t+h-s)f(s, u(s), u(b_1(s)), u(b_2(s)), \dots, u(b_m(s)))\, ds \\ & - \int_{t_0}^{t} T\,(t-s)f(s, u(s), u(b_1(s)), u(b_2(s)), \dots, u(b_m(s)))\, ds \\ & + \sum_{t_0 \le \tau_k < t+h} T\,(t+h-\tau_k)I_k\big(u(\tau_k)\big) - \sum_{t_0 \le \tau_k < t} T\,(t-\tau_k)I_k\big(u(\tau_k)\big). \end{aligned}$$

Consequently, by applying Gronwall's inequality we have

$$\| \, u(t+h) - u(t) \, \| \le (C_* h + MC_3 i(t, t+h)) \exp\,(\,aML(1+mK)).$$

Thus $\| \, u(t+h) - u(t) \, \| \to 0$as $h \to 0$,and u is Lipschitz continuous on each interval of continuity in I. The Lipschitz continuity of u on each interval of continuity in I combined with the Lipschitz continuity of f on $I \times E^{m+1}$ imply that the mapping $t \mapsto f(t, u(t), u(b_1(t)), .., u(b_m(t)$ is Lipschitz continuous on each interval of continuity in I. This property, together with the assumptions of Theorem 2, implies that the linear Cauchy problem

$$\begin{aligned} &\dot{v}(t) + Av(t) = f(t, u(t), u(b_1(t)), \dots, u(b_m(t))), \\ &\qquad\qquad t \in I, \ \ t \ne \tau_k, \\ &v(\tau_k + 0) = u(\tau_k) + I_k\big(u(\tau_k)\big), \quad k = 1,2,\dots,\kappa, \\ &v(t_0) = u_0 - g(u), \end{aligned}$$

has a unique classical solution v such that

$$v(t) = T(t-t_0)u_0 - T(t-t_0)g(u) + \int_{t_0}^{t} T(t-s)f(s, u(s), u(b_1(s)), \dots, u(b_m(s)))\, ds + \sum_{t_0 \le \tau_k < t} T\,(t-\tau_k)I_k(u(\tau_k)), \quad t \in I.$$

Consequently, u is the unique classical solution of the nonlocal impulsive Cauchy problem (1) (details of the proof in [6]).

III. Continuous Dependence of a Mild Solution on the Initial Condition

Theorem3. *Supposethat the functions f, g and $I(u)$satisfy* *theassumptions* ***A1*** *–* ***A3*** *and there exist constants* μ_1, μ_2, μ_3 *such that*

1) $\| g(u) - g(\tilde{u}) \| \leq \mu_1 \| u - \tilde{u} \|$,
2) $\| f(s, u(s), \ldots, u(b_m(s))) - f(s, \tilde{u}(s), \ldots, \tilde{u}(b_m(s))) \| \leq \mu_2 \| u - \tilde{u} \|$,
3) $\| I_k(u(\tau_k)) - I_k(\tilde{u}(\tau_k)) \| \leq \mu_3 \| u(\tau_k) - \tilde{u}(\tau_k) \|$,

where $u, \tilde{u} \in C(I, E)$. *If* u *and* $\tilde{u}$ *are mild solutions of theproblem(1)with the respective initial values* $u_0, \tilde{u}_0$ *and theconstants* μ_1 *and* $\mu = max\{\mu_2, \mu_3\}$ *satisfy the inequality*

$$\mu_1 < \frac{exp(-(t_0 + a)M\mu)(1 + M\mu)^{-\kappa}}{M},$$

then the following inequality holds[6]*:*

$$\|u(t) - \tilde{u}(t)\| \leq \frac{M\,exp\big((t_0 + a)M\mu\big)(1 + M\mu)^{\kappa}}{1 - M\mu_1\,exp\big((t_0 + a)M\mu\big)(1 + M\mu)^{\kappa}} \|u_0 - \tilde{u}_0\|. \quad (7)$$

Proof. Assume that $u, \tilde{u}$ are mild solutions of problem (1). Then

$$u(t) - \tilde{u}(t) = [T(t - t_0)(u_0 - \tilde{u}_0) - T(t - t_0)\big(g(u) - g(\tilde{u})\big)] + \int_{t_0}^{t} T(t - s)\Big[f\Big(s, u(s), u\big(b_1(s)\big), \ldots, u\big(b_m(s)\big)\Big) - f\Big(s, \tilde{u}(s), \tilde{u}\big(b_1(s)\big), \ldots, \tilde{u}\big(b_m(s)\big)\Big)\Big]\, ds + \sum_{t_0 \leq \tau_k < t+h} T(t - \tau_k)[I_k(u(\tau_k)) - I_k(\tilde{u}(\tau_k))],$$

where $t \in [t_0, t_0 + a]$. From **A1**–**A3** and the hypotheses of the theorem, and applying Gronwall's inequality we have

$$\|u(t) - \tilde{u}(t)\| \leq \{\| u_0 - \tilde{u}_0 \| + \mu_1 \| u - \tilde{u} \|\} M \exp\big((t_0 + a)M\mu\big)(1 + M\mu)^{\kappa}.$$

We can also write this inequality in the form

$$[1 - M\mu_1 \exp((t_0 + a)M\mu)(1 + M\mu)^{\kappa}] \| u(t) - \tilde{u}(t) \| \leq M \exp((t_0 + a)M\mu)(1 + M\mu)^{\kappa} \| u_0 - \tilde{u}_0 \|.$$

For more details of the proof,you can see [6] and the references therein.

Remark 1.If $\mu_1 = \kappa = 0$, then inequality (7) is reduced to the classical inequality.

$$\| u(t) - \tilde{u}(t) \| \leq M \exp((t_0 + a)M\mu) \| u_0 - \tilde{u}_0 \|,$$

This is a characteristic for the continuous dependence of the semilinear functional-differential evolution Cauchy problem with the classical initial condition.

IV. Second Order Impulsive Differential Equations with Nonlocal Conditions

Consider a second order semilinear nonlocal impulsive differential equation of the form

$$\begin{cases} x''(t) = Ax(t) + f\big(t, x(t), x'(t)\big), \\ \qquad t \in J = [0, b], \quad t \neq t_k, \\ \Delta x|_{t=t_k} = I_k\big(x(t_k)\big), \quad k = 1, \ldots, m, \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k)), x'(t_k)), \;\; k = 1, \ldots, m, \\ x(0) + g(x) = x_0, \quad x'(0) = x_1, \end{cases} \quad (8)$$

where A is a linear operator from a real Banach space X into itself with the norm $\|\cdot\|$ and

$$x_0, x_1 \in X,\; 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b,$$
$$\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-),\; \Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-),$$

and f, g, I_k, and $\bar{I}_k$ $(k = 1, 2, \ldots, m)$ are given functions to be specified. Let A be the same linear operator as in (1).

We need the following assumptions[9], [10]:

H1: The operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t): t \in \mathbb{R}\}$ of bounded linear operators from X into itself defined by

$$Ax := \frac{d^2}{dt^2} C(t)x|_{t=0}, \qquad x \in D(A), A: X \supset D(A) \to X.$$

The associated sine family $\{S(t): t \in \mathbb{R}\}$ is defined by

$$S(t)x := \int_0^t C(s)x\, ds, \quad x \in X, t \in \mathbb{R}.$$

H2: The adjoint operator A^* is densely defined in X^*, that is,$\overline{D(A^*)} = X^*$. It can be easily seen from these assumption that for some constant $M \geq 1$ and $\omega \geq 0$ we have

$$\| C(t) \| \leq Me^{\omega|t|} \text{ and } \| S(t) \| \leq Me^{\omega|t|}, \text{ for all } t \in \mathbb{R}.$$

Denote $J_0 = [0, t_1], J_k = (t_k, t_{k+1}],\; J' = J \backslash \{t_k\},\; k = 1, 2, \ldots, m$, and define the class of functions $PC(J, X) = \{x: J \to X, x \in C(J_k, X),\; k = 0, 1, \ldots, m$ and there exist $x(t_k^+), x(t_k^-),\; k = 1, 2, \ldots, m$,with $x(t_k) = x(t_k^-)\}$ and $PC^1(J, X) = \{x \in PC(J, X): x' \in C(J_k, X), k = 0, 1, 2, \ldots, m$ and there exist $x'(t_k^+), x'(t_k^-), k = 1, 2, \ldots, m$, with $x'(t_k) = x'(t_k^-)\}$. It is clear that $PC(J, X)$ and $PC^1(J, X)$are Banach spaces with respective norms

$$\| x \|_{PC} = \max\left\{\sup_{s \in J_k} \|x(s)\|,\; k = 0, 1, .., m\right\},$$
$$\| x \|_{PC^1} = \max\{\| x \|_{PC}, \| x' \|_{PC}\}.$$

Definition 5[12].A map $f: j \times X \times X \to X$ is said to be L^1 *-Carathéodory* if $f: (\cdot, \omega, v): J \to X$ is measurable for every $\omega, v, \in X$, $f: (t, \cdot, \cdot): X \times X \to X$ is continues for almost all $t \in J$ and for each $i > 0$ there exists $\alpha_i \in L^1(J, \mathbb{R}_+)$ such that for almost all $t \in J$.

$$\sup_{\|\omega\|, \|v\| \leq 1} \| f(t, \omega, v) \| \leq \alpha_i(t).$$

A function $x \in PC^1(J, X) \cap C^2(J', X)$ satisfying $x(t) \in D(A)$ for all $t \in J$ and (8) is called a*classicalsolution* of the system (8). Then we get

$$x'(t) = C(t)x'(0) + \int_0^t f\big(s, x(s), x'(s)\big)ds + \sum_{0 < t_k < t} [x'(t_k^+) - x'(t_k)],\; \forall t \in J, \quad (9)$$

and, therefore,

$$x(t) = C(t)x_0 + S(t)x_1 + x_0 + x_1 t + \sum_{0 < t_k < t} [x(t_k^+) - x(t_k)] + \sum_{0 < t_k < t} [x'(t_k^+) - x'(t_k)](t - t_k) + \int_0^t S(t - s) f\big(s, x(s), x'(s)\big)\, ds,\; \forall t \in J. \quad (10)$$

In fact, let us assume $t_k < t < t_{k+1}$(here $t_0 = 0, t_{m+1} = b$).Using the successive iteration techniques on the intervals $[0, t_1], [t_1, t_2], \ldots, [t_k^+, t]$ we have

$$x'(t_1) - x'(0) = \int_0^{t_1} f\left(s, x(s), x'(s)\right) ds,$$

$$x'(t_2) - x'(t_1^+) = \int_{t_1}^{t_2} f\,(s, x(s), x'(s)) ds,$$

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$$x'(t) - x'(t_k^+) = \int_{t_k^+}^{t} f\left(s, x(s), x'(s)\right) ds.$$

Adding these, we get

$$x'(t) = x'(0) + \int_0^t f\left(s, x(s), x'(s)\right) ds + \sum_{0<t_k<t} [x'(t_k^+) - x'(t_k)].$$

Similarly, we can get

$$x(t) = x(0) + \int_0^t x'(s) ds + \sum_{0<t_k<t} [x(t_k^+) - x(t_k)] \quad (11)$$

and, using the equation (9) in(11), we get the result (10).

Assume the following additional assumptions are satisfied[12]:

H3: $f: J \times X \times X \to X$ is an L^1-Carathéodorymapping.

H4: $I_k \in C(\cdot, X)$, $\bar{I}_k \in C(X \times X, X)$ and there are constants $d_k, \bar{d}_k$ such that $\| I_k(\omega_k) \| \le d_k$, $\| \bar{I}_k(\omega, v) \| \le \bar{d}_k$, $k = 1,2, \ldots, m$ for every $\omega, v \in X$.

H5: $g: PC(J, X) \to X$ is a continuous function and for some constant M, $\| g(x) \| \le M$, for every $x \in PC(J, X)$.

H6: There exists a function $p \in L^1(J, \mathbb{R}_+)$ such that $\| f(t, \omega, v) \| \le p(t)\psi(\| \omega \| + \| v \|)$ fora.e. $t \in J$ and every $\omega, v \in X$ where $\psi: [0, \infty) \to [0, \infty)$ is anondecreasing continuous function satisfying the inequality

$$(b+1)\int_0^b p\,(s) ds < \int_c^\infty \frac{ds}{\psi(s)}$$

where the constant c can be determined as

$$c = \|x_0\| + M + (b+1)\|x_1\| + \sum_{k=1}^{m} \left[d_k + (b + 1 - t_k)\bar{d}_k\right]$$

Definition 6. Consider the following second order nonlocal problem [10]

$$\begin{cases} x''(t) = Ax(t) + f(t, x(t), x'(t)), t \in (0, T] \\ x(0) = x_0, \;\; x: [0, T] \to X, \;\; f: [0, T] \times X^2 \to X \\ x'(0) + \displaystyle\sum_{i=1}^{p} h_i\, x(t_i) = x_1, \quad x_0, x_1 \in X, \;\; h_i \in \mathbb{R}, \\ (i = 1,2, \ldots, p), 0 < t_1 < t_2 < \cdots < t_p \le T \end{cases} \quad (12)$$

Then the function $x \in C^1((0, T], X)$ satisfying the integral equation

$$\begin{cases} x(t) = C(t)x_0 S(t)x_1 - S(t)\left(\displaystyle\sum_{i=1}^{p} h_i\, x(t_i)\right) \\ + \displaystyle\int_0^t S(t-s) f\left(s, x(s), x'(s)\right) ds, \;\; s \in [0, T] \end{cases}$$

is said to be a mild solution of the nonlocal Cauchy problem (12).

Theorem 4.*Under the assumptions**H3-H6**the secondorder impulsive problem with nonlocal conditions(8)has at least one solution on the interval J.*

Proof. Define the space $B = PC^1(J, X)$.We can show that the functional operator G defined by

$$Gx(t) = C(t)(x_0 - g(x)) + tS(t)x_1 + \sum_{0<t_k<t} I_k\,(x(t_k)) + \sum_{0<t_k<t} \bar{I}_k\,(x(t_k), x'(t_k))(t - t_k) + \int_0^t (t-s) f(s, x(s), x'(s)) ds, \; t \in J,$$

has a fixed point and this fixed point is the solution of the equation (8). For details of the proof, you can see [12] and the given references. It can be shown that $G: B \to B$ is a completely continuous operator. Let $B_i = \{x \in B: \| x \|_{PC^1} \le i\}$, for some $i > 1$. G maps B_i into an equicontinuous family. To show this result,let $x \in B_i$, and $t, \bar{t} \in J$ satisfy $0 < t < \bar{t} \le b$.Then we have

$$\begin{aligned} &\| (Gx)(t) - (Gx)(\bar{t}) \| \le (\bar{t} - t) \| x_1 \| \\ &+ \sum_{t \le t_k \le \bar{t}} d_k + \sum_{0<t_k<t} (\bar{t} - t)\bar{d}_k + \sum_{t \le t_k < \bar{t}} (\bar{t} - t_k)\bar{d}_k \\ &+ \int_0^t (\bar{t} - t)\alpha_i(s)\, ds + \int_t^{\bar{t}} (\bar{t} - s)\alpha_i(s)\, ds. \end{aligned} \quad (13)$$

Similarly,

$$\begin{aligned} &\left\|(Gx)'(t) - (Gx)'(\bar{t})\right\| \\ &\le \left\| \int_0^t f\,(s, x(s), x'(s)) ds - \int_0^{\bar{t}} f\,(s, x(s) x'(s)) ds \right\| \\ &+ \left\| \sum_{0<t_k<t} \bar{I}_k\left(x(t_k), x'(t_k)\right) - \sum_{0<t_k<\bar{t}} \bar{I}_k\,(x(t_k), x'(t_k)) \right\| \\ &\le \int_t^{\bar{t}} \alpha_i\,(s) ds + \sum_{t \le t_k < \bar{t}} \bar{d}_k. \end{aligned} \quad (14)$$

The right-hand sides of(13) and (14) are independent of x and approach zero when$\bar{t} \to t$, this means G maps B_i into an equicontinuous family of functions [12]. By using the Arzelà-Ascoli theorem $G: B \to B$ is compact. It can be easily shown that $G: B \to B$ is continuous. This completes the proof of the theorem.

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