On Improving the Semi-Local Convergence of Newton-Type Projection Method for Ill-Posed Hammerstein Type Operator Equations

Monnanda Erappa Shobha

***Abstract*—A Two Step Newton-Tikhonov Projection method is presented for obtaining a stable approximate solution of nonlinear ill-posed Hammerstein type operator equations *KF*(*x*) =*f*. The regularization parameter is chosen according to the adaptive parameter choice strategy suggested by Perverzev and Schock (2005). The error estimates obtained with respect to the general source conditions are of optimal order. We also give the numerical example which confirms the efficiency of the proposed method.**

***Index Terms*—Adaptive method, discretized newton tikhonov method, hammerstein operators, monotone operator, regularization.**

I. Introduction

This paper deals with the finite dimensional realization of a method considered in [1] for (nonlinear) Hammerstein-type equation ([2]-[4])

$$KF(x) = f \tag{1}$$

Here $F : D(f) \subseteq X \to X$ is nonlinear, $K : X \to Y$ is a bounded linear operator ([2], [3]) and X(real), Y are Hilbert spaces with inner product $\langle .,. \rangle$ and norm $\|.\|$ respectively. It is assumed that $f^{\delta} \in Y$ are the available noisy data with $\|f - f^{\delta}\| \le \delta$.

The aim is to approximate the x_0-minimum norm solution (x_0-MNS) $\hat{x}$ of (1). Recall that [3], [4], $\hat{x}$ is said to be an x_0 -MNS if $\|F(\hat{x}) - F(x_0)\| = \min\{\|F(x) - F(x_0)\| : KF(x) = f, x \in D(F)\}$

In [1] we considered two cases of F; in the first case we assume that $F'(x)^{-1}$ exist and in the second case we assume F is a monotone operator (i.e., $\langle F(x) - F(y), x - y \rangle \ge 0, \forall x, y \in D(F)$) and $F'(x)^{-1}$ does not exist. The derived error estimates in [1] was of optimal order and we obtained quartic convergence. In this paper we consider the finite dimensional realization of the second case i.e., F is monotone but $F'(x)^{-1}$ doesnot exist.

The organization of this paper is as follows. Preliminaries are given in Section II. In Section III we investigate the semi-local convergence (i.e., convergence of the iterations in a ball of radius r centered at x_0) of the proposed Discretized Two Step Newton Tikhonov Method(DTSNTM). Section VI gives the algorithm and Section V deals with the implementation of the method and a numerical example which confirms the efficiency of the proposed method. And finally paper ends with a conclusion in Section 6.

II. Preliminaries

In this section we state the results of Section II in [5], needed for this paper. As in [5] we assume that F possess a uniformly bounded Frechet derivative for each $x \in D(F)$, i.e.,$\|F'(x)\| \le M, \forall x \in D(F)$ for some M.

Let $\{P_h\}_{h \ge 0}$ be a family of orthogonal projections on X.

Let $\varepsilon_h := \|K(I - P_h)\|$, $\tau_h := \|F'(x)(I - P_h)\|$, $\forall x \in D(F)$ and $\{b_h : h > 0\}$ is such that $\lim_{h\to 0} \frac{\|(I - P_h)x_0\|}{b_h} = 0$, $\lim_{h\to 0} \frac{\|(I - P_h)F(x_0)\|}{b_h} = 0$ *and* $\lim_{h\to 0} b_h = 0$.

We assume that $\varepsilon_h \to 0$ and $\tau_h \to 0$ as $h \to 0$.

The above assumption is satisfied if, $P_h \to I$ point wise and if K and F'(x) are compact operators. Further we assume that $\varepsilon_h < \varepsilon_0$, $\tau_h \le \tau_0$, $b_h \le b_0$ and $\delta \in (0, \delta_0]$ where $\delta_0 + \varepsilon_0 < (\frac{2}{2M+3})\sqrt{\alpha_0}$.

Further as in [1], we solve (1) for x by first solving

$$Kz=f \tag{2}$$

for *z* and then solving the non-linear problem

$$F(x)=z \tag{3}$$

The discretized Tikhonov regularization method for the regularized equation (2) with f^{δ} in place of f, consists of solving the equation

$$(P_h K^* K P_h + \alpha P_h)(z_{\alpha}^{h,\delta} - P_h F(x_0)) = P_h K^* [f^{\delta} - KF(x_0)] \tag{4}$$

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Monnanda Erappa Shobha is with the Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, India-575025 (e-mail: shobha.me@gmail.com.).

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The following assumption is used as in [1], [3] to obtain the error estimate.

Assumption 2.1: There exists a continuous, strictly monotonically increasing function $\varphi:(0,a]\to(0,\infty)$ with $a\ge\|K\|^2$ satisfying;

- $\lim_{\lambda\to 0}\varphi(\lambda)=0$
- $\sup_{\lambda>0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha}\le\varphi(\alpha)\quad\forall\lambda\in(0,a],$ *and*
- there exists $v\in X,\|v\|\le 1$ such that

$$F(\hat{x})-F(x_0)=\varphi(K^*K)v\ .$$

Theorem 2.2: (see [5], Theorem 2.4) Suppose Assumption 2.1 holds. Let $z_\alpha^{h,\delta}$ be as in (4) and $b_h\le\frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$.

Then

$$\|F(\hat{x})-z_\alpha^{h,\delta}\|\le C(\varphi(\alpha)+(\frac{\delta+\varepsilon_h}{\sqrt{\alpha}})),\tag{5}$$

where $C=\frac{1}{2}\max\{M\rho,1\}+1\cdot$

A. A Priori Choice of the Parameter

Note that the estimate $\varphi(\alpha)+\frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$ in Theorem 2.2 is of optimal order for the choice $\alpha:=\alpha(\delta,h)$ which satisfies $\varphi(\alpha(\delta,h))=\frac{\delta+\varepsilon_h}{\sqrt{\alpha(\delta,h)}}$. Let $\psi(\lambda):=\lambda\sqrt{\varphi^{-1}(\lambda)},0<\lambda\le a.$

Then we have $\delta+\varepsilon_h=\sqrt{\alpha(\delta,h)}\varphi(\alpha(\delta,h))=\psi(\varphi(\alpha(\delta,h)))$ and $\alpha(\delta,h)=\varphi^{-1}(\psi^{-1}(\delta+\varepsilon_h))$. So the relation (1.5) leads to $\|F(\hat{x})-z_\alpha^{h,\delta}\|\le 2C\psi^{-1}(\delta+\varepsilon_h)$.

B. An Adaptive Choice of the Parameter

In this subsection, we consider the adaptive method introduced by Pereverzev and Shock [6] for choosing the parameter α. Let $D_N=\{\alpha_i:0<\alpha_0<\alpha_1<\alpha_2<\cdots<\alpha_N\}$ be the set of possible values of the parameter α.

Let

$$l:=\max\{i:\varphi(\alpha_i)\le\frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}}\}<N,\tag{6}$$

$$k=\max\{i:\alpha_i\in D_N^+\}\tag{7}$$

where

$$D_N^+=\{\alpha_i\in D_N:\|z_{\alpha_i}^\delta-z_{\alpha_j}^\delta\|\le\frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}},j=0,1,2,....,i-1\}.$$

Theorem 2.3: (cf. [5] , Theorem 2.5) Let l be as in (6) *k be as in (7) and* $z_{\alpha_k}^{h,\delta}$ *be as in (4) with* $\alpha=\alpha_k$. *Then* $l\le k$ *and* $\|F(\hat{x})-z_{\alpha_k}^{h,\delta}\|\le C(2+\frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta+\varepsilon_h)$.

III. Convergence Analysis of DTSNTM

Let $B(x_0,r)$ denotes the ball of radius r with center x_0.

Assumption 3.1 (cf. [7], Assumption 3 (A3)) There exists a constant $k_0\ge 0$ such that for every $x,u\in B(x_0,r)\cup B(\hat{x},r)\subseteq D(F)$ and $v\in X$ there exists an element $\Phi(x,u,v)\in X$ such that

$$[F'(x)-F'(u)]v=F'(u)\Phi(x,u,v),\|\Phi(x,u,v)\|\le k_0\|v\|\|x-u\|.$$

The DTSNTM is defined as:

$$y_{n,\alpha_k}^{h,\delta}=x_{n,\alpha_k}^{h,\delta}-R(x_{n,\alpha_k}^{h,\delta})^{-1}P_h[F(x_{n,\alpha_k}^{h,\delta})-z_{\alpha_k}^{h,\delta}+\frac{\alpha_k}{c}(x_{n,\alpha_k}^{h,\delta}-x_{0,\alpha_k}^{h,\delta})]\tag{8}$$

And

$$x_{n+1,\alpha_k}^{h,\delta}=y_{n,\alpha_k}^{h,\delta}-R(y_{n,\alpha_k}^{h,\delta})^{-1}P_h[F(y_{n,\alpha_k}^{h,\delta})-z_{\alpha_k}^{h,\delta}+\frac{\alpha_k}{c}(y_{n,\alpha_k}^{h,\delta}-x_{0,\alpha_k}^{h,\delta})]\tag{9}$$

where $x_{0,\alpha_k}^{h,\delta}:=P_hx_0$ is the initial guess, $c\le\alpha_k$, $R(x):=P_hF'(x)P_h+\frac{\alpha_k}{c}P_h$. First we show that $(x_{n,\alpha_k}^{h,\delta})$ converges to the zero $x_{c,\alpha_k}^{h,\delta}$ of

$$P_h(F(x)+\frac{\alpha_k}{c}(x-x_0))=P_hz_{\alpha_k}^{h,\delta}\tag{10}$$

and then we show that $x_{c,\alpha_k}^{h,\delta}$ is an approximation to the solution $\hat{x}$ of (1.1). Let

$$e_{n,\alpha_k}^{h,\delta}:=\|y_{n,\alpha_k}^{h,\delta}-x_{n,\alpha_k}^{h,\delta}\|,\forall n\ge 0.\tag{11}$$

and let k_0 be such that $k_0<\min\{1,\frac{2}{3(1+\tau_0)}\}$. Let $g:(0,1)\to(0,1)$ be the function defined by

$$g(t)=\frac{27k_0^3}{8}(1+\tau_0)^3t^3\ \ \forall t\in(0,1).\tag{12}$$

Further let $\|\hat{x}-x_0\|\le\rho$, with $\rho<\frac{1}{M}(1-(\frac{3}{2}+M)\frac{\delta_0+\varepsilon_0}{\sqrt{\alpha_0}})$ and $\gamma_\rho:=M\rho+(\frac{3}{2}+M)(\frac{\varepsilon_0+\delta_0}{\sqrt{\alpha_0}})$.

Theorem 3.2: Let $e_{n,\alpha_k}^{h,\delta}$ and g be as in equation (11) and (12) respectively, $x_{n,\alpha_k}^{h,\delta}$ and $y_{n,\alpha_k}^{h,\delta}$ be as in (9) and (8) respectively with $\delta\in(0,\delta_0]$, $\alpha=\alpha_k$ and $\varepsilon_h\in(0,\varepsilon_0]$.

Then the following hold:

a) $\|x_{n,\alpha_k}^{h,\delta}-y_{n-1,\alpha_k}^{h,\delta}\|\le(1+\tau_0)\frac{3k_0e_{n-1,\alpha_k}^{h,\delta}}{2}\|y_{n-1,\alpha_k}^{h,\delta}-x_{n-1,\alpha_k}^{h,\delta}\|;$

b) $\|x_{n,\alpha_k}^{h,\delta}-x_{n-1,\alpha_k}^{h,\delta}\|\le(1+(1+\tau_0)\frac{3k_0e_{n-1,\alpha_k}^{h,\delta}}{2})\|y_{n-1,\alpha_k}^{h,\delta}-x_{n-1,\alpha_k}^{h,\delta}\|;$

c) $\|y_{n,\alpha_k}^{h,\delta}-x_{n,\alpha_k}^{h,\delta}\|\le g(e_{n-1,\alpha_k}^{h,\delta})\|y_{n-1,\alpha_k}^{h,\delta}-x_{n-1,\alpha_k}^{h,\delta}\|;$

d) $g(e^{h,\delta}_{n,\alpha_k}) \le g(\gamma_\rho)^{4^n}$, $\forall n \ge 0$;

e) $e^{h,\delta}_{n,\alpha_k} \le g(\gamma_\rho)^{(4^n-1)/2}\gamma_\rho, \forall n \ge 0.$

Proof. Proofs of a), b) and c) are analogous to the proof of corresponding results of Theorem 3.4 in [5]. Further, since for $\mu \in (0,1)$, $g(\mu t) \le \mu^3 g(t)$, for all $t \in (0,1)$, by c) we have, $g(e^{h,\delta}_{n,\alpha_k}) \le g(e^{h,\delta}_{0,\alpha_k})^{4^n}$ and $e^{h,\delta}_{n,\alpha_k} \le g^4(e^{h,\delta}_{n-2,\alpha_k})e^{h,\delta}_{n-1,\alpha_k}$

$$\begin{aligned}&\le g^4(e^{h,\delta}_{n-2,\alpha_k})g^4(e^{h,\delta}_{n-3,\alpha_k})e^{h,\delta}_{n-2,\alpha_k}\cdots g(e^{h,\delta}_{0,\alpha_k})e^{h,\delta}_{0,\alpha_k}\\ &\le g(e^{h,\delta}_{0,\alpha_k})^{4^{n-1}+4^{n-2}+\cdots+1}e^{h,\delta}_{0,\alpha_k}\\ &\le g(e^{h,\delta}_{0,\alpha_k})^{(4^n-1)/2}e^{h,\delta}_{0,\alpha_k}\end{aligned}\tag{13}$$

provided $e^{h,\delta}_{n,\alpha_k} < 1, \forall n \ge 0.$ From (13) it is clear that, $e^{h,\delta}_{n,\alpha_k} \le 1$ if $e^{h,\delta}_{0,\alpha_k} \le 1.$ Now since g is monotonic increasing and $e^{h,\delta}_{0,\alpha_k} \le \gamma_\rho$, (see [5], equation (3.28)) we have $g(e^{h,\delta}_{0,\alpha_k}) \le g(\gamma_\rho).$ This proves d) and e).

Theorem 3.3: Let $r = (\frac{1}{1-g(\gamma_\rho)} + (1+\tau_0)\frac{3k_0}{2}\frac{\gamma_\rho}{1-g(\gamma_\rho)^2})\gamma_\rho$ and the assumptions of Theorem 3.2 hold. Then $x^{h,\delta}_{n,\alpha_k}, y^{h,\delta}_{n,\alpha_k} \in B_r(P_h x_0),$ for all $n \ge 0.$

Proof. Analogous to the proof of Theorem 3.5 in [5]. The main result of this section is the following Theorem.

Theorem 3.4: Let $y^{h,\delta}_{n,\alpha_k}$ and $x^{h,\delta}_{n,\alpha_k}$ be as in (8) and (9) respectively and assumptions of Theorem 3.3 hold. Then $(x^{h,\delta}_{n,\alpha_k})$ is a Cauchy sequence in $B_r(P_h x_0)$ and converges to $x^{h,\delta}_{c,\alpha_k} \in \overline{B_r(P_h x_0)}.$ Further $P_h[F(x^{h,\delta}_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^{h,\delta}_{c,\alpha_k} - x_0)] = P_h z^{h,\delta}_{\alpha_k}$ and $\| x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{c,\alpha_k} \| \le \overline{C_0} e^{-\gamma_1 4^n}$,

where

$\overline{C_0} = (\frac{1}{1-g(\gamma_\rho)^4} + (1+\tau_0)\frac{3k_0\gamma_\rho}{2}\frac{1}{1-(g(\gamma_\rho)^2)^4}g(\gamma_\rho)^{4^n})\gamma_\rho$ and $\gamma_1 = -\log g(\gamma_\rho).$

Proof. Analogous to the proof of Theorem 3.6 in [5]

The following Assumption is used in the further analysis.

Assumption 3.5: There exists a continuous, strictly monotonically increasing function $\varphi_1 : (0,b] \to (0,\infty)$ with $b \ge \| F'(x_0) \|$ satisfying;

- $\lim_{\lambda \to 0} \varphi_1(\lambda) = 0,$
- $\sup_{\lambda \ge 0} \frac{\alpha\varphi_1(\lambda)}{\lambda+\alpha} \le \varphi_1(\alpha)$, $\forall \lambda \in (0,b]$ and
- There exists $v \in X$ with $\| v \| \le 1$ (cf. [8]) such that

$$x_0 - \hat{x} = \varphi_1(F'(x_0))v.$$

- For each $x \in B_r(x_0) := \{x : \| x - x_0 \| < r\}$ there exists a bounded linear operator $G(x, x_0)$ (cf. [9]) such that

$$F'(x) = F'(x_0)G(x,x_0) \text{ with } \| G(x,x_0) \| \le K_1.$$

Hereafter we assume that $r < \frac{1}{k_0}$ and $K_1 < \frac{1-k_0 r}{1-c}.$

Theorem 3.6 (see [5]}, Theorem 3.7) Suppose x^{δ}_{c,α_k} is the solution of $F(x) + \frac{\alpha_k}{c}(x - x_0) = z^{\delta}_{\alpha_k}$ and Assumption 3.1 and 3.5 hold. Then

$$\| \hat{x} - x^{\delta}_{c,\alpha_k} \| \le \frac{\varphi_1(\alpha_k) + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta+\varepsilon_h)}{1-(1-c)K_1 - k_0 r}.$$

Theorem 3.7 (see [5], Theorem 3.8) Suppose $x^{h,\delta}_{c,\alpha_k}$ is the solution of (10) and Assumption 2.1 and Theorem 3.6 hold. In addition if $\tau_0 < 1,$ then

$$\| x^{h,\delta}_{c,\alpha_k} - x^{\delta}_{c,\alpha_k} \| \le \frac{2}{1-\tau_0}(\frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}).$$

The following Theorem is a consequence of Theorem 3.4, Theorem 3.6 and Theorem 3.7.

Theorem 3.8: Let $x^{h,\delta}_{n,\alpha_k}$ be as in (9), assumptions in Theorem 3.4, Theorem 3.6 and Theorem 3.7 hold. Then

$$\| \hat{x} - x^{h,\delta}_{n,\alpha_k} \| \le \overline{C_1} e^{-\gamma_1 4^n} + \overline{C_1} + \frac{2}{1-\tau_0}(\frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}})$$

where $\overline{C_1} = \frac{\varphi_1(\alpha_k) + (2+\frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta+\varepsilon_h)}{1-(1-c)K_1-k_0 r}$, and $\overline{C_0}$ & γ_1 are as in Theorem 3.4.

Theorem 3.9: Let $x^{h,\delta}_{n,\alpha_k}$ be as in (9) and assumptions in Theorem 3.8 hold. Further let $\varphi_1(\alpha_k) \le \varphi(\alpha_k)$ and

$$n_k := \min\{n : e^{-\gamma_1 4^n} \le \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}.$$

Then $\| \hat{x} - x^{h,\delta}_{n,\alpha_k} \| = O(\psi^{-1}(\delta+\varepsilon_h)).$

IV. Algorithm

Note that for $i, j \in \{0,\ 1,\ 2,\ \cdots,\ N\},$ $z^{h,\delta}_{\alpha_i} - z^{h,\delta}_{\alpha_j} = (\alpha_j - \alpha_i)(P_h K^* K P_h + \alpha_j I)^{-1}(P_h K^* K P_h + \alpha_i I)^{-1} P_h K^*(f^\delta - KF(x_0))$.

Therefore the balancing principle algorithm associated with the choice of the parameter specified in Section II involves the following steps.

- Choose α_0 such that $\delta_0 + \varepsilon_0 < \frac{2\sqrt{\alpha_0}}{2M+3}$ and $\mu > 1;$
- $\alpha_i = \mu^{2i}\alpha_0;$
- solve for w_i

$$(P_h K^* K P_h + \alpha_i I)w_i = P_h K^*(f^\delta - KF(x_0)); \tag{14}$$

- Solve for $j < i$,

$$z_{ij}^{h,\delta} : (P_h K^* K P_h + \alpha_j I) z_{ij}^{h,\delta} = (\alpha_j - \alpha_i) w_i; \qquad (15)$$

- If $\| z_{ij}^{h,\delta} \| > \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}$, then take $k = i-1$;
- Otherwise, repeat with $i+1$ in place of i.
- Choose $n_k = \min\{n : e^{-\gamma_1 4^n} \le \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$
- Solve $x_{n_k,\alpha_k}^{h,\delta}$ using the iteration (9).

V. Implementation of the Method

In this section we present an example for implementing the algorithm mentioned in above section. We take an example (see [7] , section 4.3.) satisfying the assumptions made in this paper. We consider the operator $KF : L^2(0,1) \to L^2(0,1)$ where $K : L^2(0,1) \to L^2(0,1)$ defined by $K(x)(t) = \int_0^1 k(t,s)x(s)ds$ and $F : D(F) \subseteq L^2(0,1) \to L^2(0,1)$ defined by $F(u) := \int_0^1 k(t,s)u^3(s)ds$,
where

$$k(t,s) = \begin{cases} (1-t)s, 0 \le s \le t \le 1 \\ (1-s)t, 0 \le t \le s \le 1 \end{cases}.$$

Then for all $x(t), y(t) : x(t) > y(t)$:

$$\langle F(x) - F(y), x-y \rangle = \int_0^1 \left[\int_0^1 k(t,s)(x^3-y^3)(s)ds \right](x-y)(t)dt \ge 0.$$

Thus the operator F is monotone. The Frechet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t,s)(u(s))^2 w(s)ds.$$

So for any $u \in B_r(x_0), x_0{}^2(s) \ge k_3 > 0, \forall s \in (0,1)$, we have $F'(u)w = F'(x_0)G(u,x_0)w$,
where

$$G(u,x_0) = (\frac{u}{x_0})^2.$$

Further observe that

$$[F'(v) - F'(u)]w(s) = 3\int_0^1 k(t,s)[\mathrm{v}^2(s) - u^2(s)]\mathrm{w}(s)ds$$

$$:= F'(u)\Phi(u,v,w),$$
where

$$\Phi(u,v,w) = [\frac{v^2}{u^2} - 1]\mathrm{w}.$$

Thus F satisfies the Assumption 3.1 (cf. [10], Example 2.7).

Let V_n be a sequence of finite dimensional subspaces of X and $\dim V_n = n+1$. We choose the linear splines $\{v_1, v_2, \cdots, v_{n+1}\}$ in a uniform grid of $n+1$ points in $[0, 1]$ as a basis of V_n. Let $P_h = P_{\frac{1}{n}}$ denote the orthogonal projection on X with range $R(P_h) = V_n$. We assume that $\| P_h x - x \| \to 0$ as $h \to 0$ for all $x \in X$.

Since $w_i \in V_n$, w_i can be written as $\sum_{i=1}^{n+1} \lambda_i v_i$ for some scalars $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ and w_i is a solution of (14) if and only if $\bar{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_{n+1})^T$ is the unique solution of $(M_n + \alpha_i B_n)\bar{\lambda} = \bar{a}$ where
$M_n = (\langle Kv_i, Kv_j \rangle), i, j = 1, 2, \cdots, n+1$
$B_n = (\langle v_i, v_j \rangle), i, j = 1, 2, \cdots, n+1$ and
$\bar{a} = (\langle P_h K^*(f^\delta - KF(x_0)), v_i \rangle)^T, i = 1, 2, \cdots, n+1.$

One can see from (15) that $z_{ij}^{h,\delta} \in V_n$ and hence $z_{ij}^{h,\delta} = \sum_{k=1}^{n+1} \mu_k^{ij} v_k$ for some $\mu_k^{ij}, k = 1, 2, \cdots, n+1$. Then, for $j < i, z_{ij}^{h,\delta}$ is a solution of $(P_h K^* K P_h + \alpha_j I) z_{ij}^{h,\delta} = (\alpha_j - \alpha_i) w_i$ if and only if $\overline{\mu^{ij}} = (\mu_1^{ij}, \mu_2^{ij}, \cdots, \mu_{n+1}^{ij})^T$ is the unique solution of $(M_n + \alpha_j B_n)\overline{\mu^{ij}} = \bar{b}$ where $\bar{b} = (\langle (\alpha_j - \alpha_i) w_i, v_i \rangle)^T$. Compute $z_{ij}^{h,\delta}$ till $\| z_{ij}^{h,\delta} \| > \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}$ and fix $k = i-1$. Now choose $n_k = \min\{n : e^{-\gamma_1 4^n} \le \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$.

Let $\xi^n = (\xi_1^n, \xi_2^n, \cdots, \xi_{n+1}^n)$, $\eta^n = (\eta_1^n, \eta_2^n, \cdots, \eta_{n+1}^n)$, $y_{n,\alpha_k}^{h,\delta} = \sum_{i=1}^{n+1} \xi_i^n v_i$ and $x_{n,\alpha_k}^{h,\delta} = \sum_{i=1}^{n+1} \eta_i^n v_i$. Then using (8) we get

$$(P_h F'(x_{n,\alpha_k}^{h,\delta}) + \frac{\alpha_k}{c}) \sum_{i=1}^{n+1} (\xi_i^n - \eta_i^n) v_i$$
$$= \sum_{i=1}^{n+1} \lambda_i v_i - \sum_{i=1}^{n+1} P_h F(x_{n,\alpha_k}^{h,\delta}) v_i + \frac{\alpha_k}{c} \sum_{i=1}^{n+1} (x_0(t_i) - \eta_i^n) v_i,$$

where $t_1, t_2, \cdots, t_{n+1}$ are the grid points.

Observe that $(y_{n,\alpha_k}^{h,\delta} - x_{n,\alpha_k}^{h,\delta})$ is a solution of (8) if and only if $\overline{(\xi^n - \eta^n)} = (\xi_1^n - \eta_1^n, \xi_2^n - \eta_2^n, \cdots, \xi_{n+1}^n - \eta_{n+1}^n)^T$ is the unique solution of $(Q_n + \frac{\alpha_k}{c} B_n)\overline{(\xi^n - \eta^n)} = B_n[\bar{\lambda} - F_{h1} + \frac{\alpha_k}{c}(X_0 - \overline{\eta^n})]$, where
$Q_n = \langle F'(x_{n,\alpha_k}^{h,\delta}) v_i, v_j \rangle, i, j = 1, 2, \cdots, n+1,$
$F_{h1} = [F(x_{n,\alpha_k}^{h,\delta})(t_1), F(x_{n,\alpha_k}^{h,\delta})(t_2), \cdots, F(x_{n,\alpha_k}^{h,\delta})(t_{n+1})]^T$ and

$X_0 = [x_0(t_1), x_0(t_2), \cdots, x_0(t_{n+1})]^T$.

Further from (9) it follows that

$$\begin{aligned}&(P_h F'(y^{h,\delta}_{n,\alpha_k}) + \frac{\alpha_k}{c})(x^{h,\delta}_{n+1,\alpha_k} - y^{h,\delta}_{n,\alpha_k}) \\ &= P_h[z^{h,\delta}_{\alpha_k} - F(y^{h,\delta}_{n,\alpha_k}) + \frac{\alpha_k}{c}(x^{h,\delta}_{0,\alpha_k} - y^{h,\delta}_{n,\alpha_k})].\end{aligned} \tag{16}$$

Thus $(x^{h,\delta}_{n+1,\alpha_k} - y^{h,\delta}_{n,\alpha_k})$ is a solution of (16) if and only if $\overline{(\eta^{n+1} - \xi^n)} = (\eta_1^{n+1} - \xi_1^n, \eta_2^{n+1} - \xi_2^n, \cdots, \eta_{n+1}^{n+1} - \xi_{n+1}^n)^T$ is the unique solution of

$$(\tilde{Q}_n + \frac{\alpha_k}{c} B_n)\overline{(\eta^{n+1} - \xi^n)} = B_n[\bar{\lambda} - F_{h2} + \frac{\alpha_k}{c}(X_0 - \overline{\xi^n})],$$

where $\tilde{Q}_n = \langle F'(y^{h,\delta}_{n,\alpha_k}) v_i, v_j \rangle, i, j = 1, 2, \cdots, n+1,$

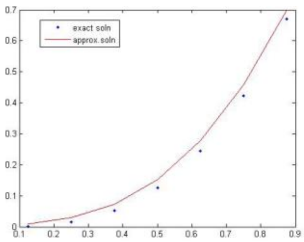
$$F_{h2} = [F(y^{h,\delta}_{n,\alpha_k})(t_1), F(y^{h,\delta}_{n,\alpha_k})(t_2), \cdots, F(y^{h,\delta}_{n,\alpha_k})(t_{n+1})]^T.$$

To illustrate the above method in our computation, we take $f(t) = \frac{1}{110}(\frac{t^{13}}{156} - \frac{t^3}{6} + \frac{25t}{156})$ and $f^\delta = f + \delta$. Then the exact solution $\hat{x}(t) = t^3$. We use $x_0(t) = t^3 + \frac{3}{56}(t - t^8)$ as our initial guess, so that the function $x_0 - \hat{x}$ satisfies the source condition $x_0 - \hat{x} = \varphi_1(F'(x_0))1$ where $\varphi_1(\lambda) = \lambda$. Thus we expect to have an accuracy of order at least $O((\delta + \epsilon_h)^{\frac{1}{2}})$.

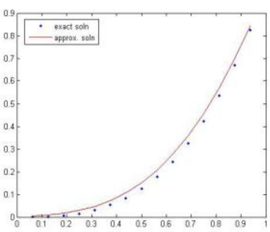
We choose $\alpha_0 = (1.3)(\delta + \epsilon_h)^2, \mu = 1.3,$ $\delta + \epsilon_h = 0.0667 = c,\ \gamma_\rho = 0.8173$ and $g(\gamma_\rho) = 0.54$ approximately. For all n the number of iteration $n_k = 1$. The results of the computation are presented in Table I. The plots of the exact and the approximate solutions obtained are given in Fig. 1.

TABLE I: ITERATIONS AND CORRESPONDING ERROR ESTIMATES

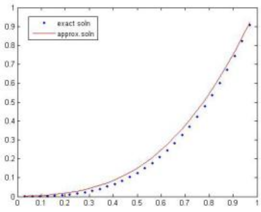
n	k	$\delta + \epsilon_h$	α	$\\| x_k^h - \hat{x} \\|$	$\frac{\\| x^h{}_k - \hat{x} \\|}{(\delta + \epsilon_h)^{1/2}}$
8	4	0.0682	0.0494	0.2390	0.9149
16	4	0.0671	0.0477	0.1809	0.6986
32	4	0.0668	0.0473	0.1391	0.5385
64	4	0.0667	0.0472	0.1111	0.4304
128	4	0.0667	0.0471	0.0936	0.3625
256	4	0.0667	0.0471	0.0833	0.3227
512	4	0.0667	0.0471	0.0776	0.3007
1024	4	0.0667	0.0471	0.0746	0.2890



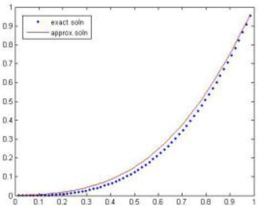
n=8



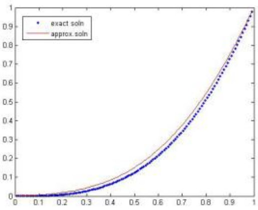
n=16



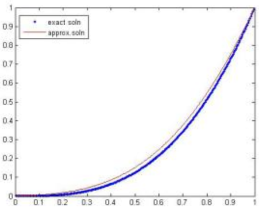
n=32



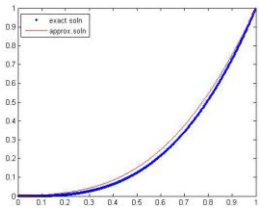
n=64



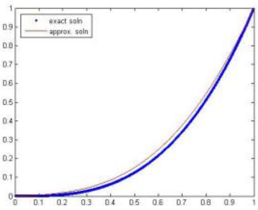
n=128



n=256



n=512



n=1024

Fig. 1. Curves of the Exact and Approximate solutions.

VI. CONCLUSION

A finite dimensional realization of a Two Step Newton-Tikhonov Projection Method is considered for obtaining an approximate solution of a nonlinear ill-posed Hammerstein type operator equation $KF(x) = f$. Under the assumption that the available data is f^δ with $\| f - f^\delta \| \leq \delta$ and the nonlinear operator F is monotone but Frechet derivative of F is non-invertible, we obtained the convergence of the method. The derived error estimates using an a priori and adaptive method of Perverzev and Schock(2005) are of optimal order with respect to a general source condition. Numerical Example presented proves the reliability of our method.

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Monnanda Erappa Shobha was born in Coorg, India on 5th July 1985. She did Masters in Mathematics from Mangalore University, Karnataka, India. She is currently a research scholar in the department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, India. Her major field of study is in the area of inverse and ill-posed problems.