On Improving the Semi-Local Convergence of Newton-Type Projection Method for Ill-Posed Hammerstein Type Operator Equations

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Abstract—A Two Step Newton-Tikhonov Projection method is presented for obtaining a stable approximate solution of nonlinear ill-posed Hammerstein type operator equations $KF(x) = f$. The regularization parameter is chosen according to the adaptive parameter choice strategy suggested by Perverzev and Schock (2005). The error estimates obtained with respect to the general source conditions are of optimal order. We also give the numerical example which confirms the efficiency of the proposed method.

Index Terms—Adaptive method, discretized newton tikhonov method, hammerstein operators, monotone operator, regularization.

I. INTRODUCTION

This paper deals with the finite dimensional realization of a method considered in [1] for (nonlinear) Hammerstein-type equation ([2]-[4])

$$KF(x) = f$$

Here $F: D(f) \subseteq X \rightarrow X$ is nonlinear, $K: X \rightarrow Y$ is a bounded linear operator ([2], [3]) and X(real), Y are Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ respectively. It is assumed that $f^\delta \in Y$ are the available noisy data with $\| f - f^\delta \| \leq \delta$.

The aim is to approximate the $x_0$-minimum norm solution $(x_0\text{-MNS})$ $x^\ast$ of (1). Recall that [3], [4], $x^\ast$ is said to be an $x_0$-MNS if

$$\| F(x^\ast) - F(x_0) \| = \min_{x \in D(F)} \| F(x) - F(x_0) \| : KF(x) = f, x \in D(F)$$

In [1] we considered two cases of $F$; in the first case we assume that $F(x)^{-1}$ exist and in the second case we assume $F$ is a monotone operator (i.e., $\langle F(x) - F(y), x - y \rangle \geq 0, \forall x, y \in D(F)$) and $F(x)^{-1}$ does not exist. The derived error estimates in [1] was of optimal order and we obtained quartic convergence. In this paper we consider the finite dimensional realization of the second case i.e., $F$ is monotone but $F'(x)^{-1}$ does not exist.

The organization of this paper is as follows. Preliminaries are given in Section II. In Section III we investigate the semi-local convergence (i.e., convergence of the iterations in a ball of radius $r$ centered at $x_0$) of the proposed Discretized Two Step Newton Tikhonov Method (DTSNTM). Section VI gives the algorithm and Section V deals with the implementation of the method and a numerical example which confirms the efficiency of the proposed method. And finally paper ends with a conclusion in Section 6.

II. PRELIMINARIES

In this section we state the results of Section II in [5], needed for this paper. As in [5] we assume that $F$ possess a uniformly bounded Frechet derivative for each $x \in D(F)$, i.e., $\| F'(x) \| \leq M, \forall x \in D(F)$ for some $M$.

Let $\{ P_h \}_{h \in H}$ be a family of orthogonal projections on X. Let $\varepsilon_h := \| K(I - P_h) \|$, $\tau_h := \| F'(x)(I - P_h) \|, \forall x \in D(F)$ and $\{ b_h : h > 0 \}$ is such that

$$\lim_{h \rightarrow 0} \frac{\| I - P_h x_0 \|}{b_h} = 0, \lim_{h \rightarrow 0} \frac{I - P_h F(x_0)}{b_h} = 0 \text{ and } \lim_{h \rightarrow 0} b_h = 0$$

We assume that $\varepsilon_h \rightarrow 0$ and $\tau_h \rightarrow 0$ as $h \rightarrow 0$.

The above assumption is satisfied if, $P_h \rightarrow I$ point wise and if $K$ and $F'(x)$ are compact operators. Further we assume that $\varepsilon_h < \varepsilon_0$, $\tau_h \leq \tau_0$, $b_h \leq b_0$ and $\delta \in (0, \delta_0)$ where $\delta_0, \varepsilon_0 < \frac{2}{2M + 3} \sqrt{\delta_0}$.

Further as in [1], we solve (1) for $x$ by first solving

$$Kz = f$$

for $z$ and then solving the non-linear problem

$$F(x) = z$$

The discretized Tikhonov regularization method for the regularized equation (2) with $f^\delta$ in place of $f$, consists of solving the equation

$$(P_h K' K P_h + \alpha P_h)(z_h^\delta - P_h F(x_0)) = P_h K'[f^\delta - KF(x_0)]$$

(4)
The following assumption is used as in [1], [3] to obtain the error estimate.

**Assumption 2.1:** There exists a continuous, strictly monotonically increasing function \( \varphi : (0, a] \to (0, \infty) \) with \( a \equiv K \| I \| \) satisfying:

- \[ \lim_{\lambda \to 0} \varphi(\lambda) = 0 \]
- \[ \sup_{\lambda > 0} \frac{\alpha \varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha) \quad \forall \lambda \in (0, a], \text{ and} \]
- there exists \( \nu \in \mathbb{X}, \| \nu \| \leq 1 \) such that

\[ F(\hat{x}) - F(x_0) = \varphi(K^*K) \nu. \]

**Theorem 2.2:** (see [5], Theorem 2.4) Suppose Assumption 2.1 holds. Let \( z^{h, \delta}_{\alpha} \) be as in (4) and \( b_{h} \leq \frac{\delta + \epsilon_{\delta}}{\sqrt{\alpha}} \).

Then

\[ \| F(\hat{x}) - z^{h, \delta}_{\alpha} \| \leq C(\varphi(\alpha) + \frac{(\delta + \epsilon_{\delta})}{\sqrt{\alpha}}), \]

where \( C = \frac{1}{2} \max \{ M \rho, 1 \} + 1 \).

**A Priori Choice of the Parameter**

Note that the estimate \( \varphi(\alpha) + \frac{\delta + \epsilon_{\delta}}{\sqrt{\alpha}} \) in Theorem 2.2 is of optimal order for the choice \( \alpha = \varphi(\delta, h) \) which satisfies

\[ \varphi(\alpha(\delta, h)) = \delta + \epsilon_{\delta} \sqrt{\varphi(\delta, h)}. \]

Let \( \psi(\lambda) = \lambda \varphi^{-1}(\lambda), 0 < \lambda \leq \alpha \).

Then we have

\[ \delta + \epsilon_{\delta} = \varphi(\varphi^{-1}(\psi(\lambda))) = \psi(\varphi(\delta, h)) \]

and \( \alpha(\delta, h) = \varphi^{-1}(\psi(\delta + \epsilon_{\delta})). \) So the relation (1.5) leads to

\[ \| F(\hat{x}) - z^{h, \delta}_{\alpha} \| \leq 2C\varphi^{-1}(\delta + \epsilon_{\delta}). \]

**B. An Adaptive Choice of the Parameter**

In this subsection, we consider the adaptive method introduced by Pereverzev and Scholz [6] for choosing the parameter \( \alpha \).

Let

\[ D_{\alpha} = \{ \alpha_1 : 0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n \} \]

be the set of possible values of the parameter \( \alpha \).

Let

\[ l = \max \{ i : \varphi(\alpha_i) \leq \frac{\delta + \epsilon_{\delta}}{\sqrt{\alpha_i}} < N \}, \]

\[ k = \max \{ i : \alpha_i \in D_{\alpha} \} \]

where

\[ D_{\alpha} = \{ \alpha_i \in D_{\alpha} : z^{h, \delta}_{\alpha_i} - z^{h, \delta}_{\alpha_j} \leq \frac{4C(\delta + \epsilon_{\delta})}{\sqrt{\alpha_i}}, j = 0, 1, 2, \ldots, i - 1 \} \]

**Theorem 2.3:** (cf. [5], Theorem 2.5) Let \( l \) be as in (6) and \( k \) be as in (7) and \( z^{h, \delta}_{\alpha} \) be as in (4) with \( \alpha = \alpha_i \). Then \( l \leq k \)

and

\[ \| F(\hat{x}) - z^{h, \delta}_{\alpha_i} \| \leq C(2 + \frac{4M}{\mu - 1}) \mu \varphi^{-1}(\delta + \epsilon_{\delta}). \]

### III. CONVERGENCE ANALYSIS OF DTSNTM

Let \( B(x_0, r) \) denotes the ball of radius \( r \) with center \( x_0 \).

**Assumption 3.1** (cf. [7], Assumption 3 (A3)) There exists a constant \( k_i \geq 0 \) such that for every \( x, u \in B(x_0, r) \cup B(\hat{x}, r) \subseteq D(F) \) and \( \nu \in \mathbb{X} \) there exists an element \( \Phi(x, u, \nu) \in \mathbb{X} \) such that

\[ F'(x) - F'(\nu) = F'(\nu)\Phi(x, u, \nu) \| \Phi(x, u, \nu) \| \leq k \| \nu - x \| \leq \| \nu - x \| \]

The DTSNTM is defined as:

\[ x_{n+1, \alpha} = x_n - R(x_{n, \alpha} - \frac{\alpha}{c} (x_{n, \alpha} - x_{0, \alpha})), \]

and then we show that \( x_{n, \alpha}^{h, \delta} \) is an approximation to the solution \( \hat{x} \) of (1.1).

Let

\[ \epsilon_{n, \alpha}^{h, \delta} := \| x_{n, \alpha}^{h, \delta} - x_{n, \alpha}^{h, \delta} \|, \forall n \geq 0. \]

and let \( k_0 \) be such that \( k_0 < \min \{ 1, \frac{2}{3(1 + r_0)} \} \). Let

\[ g : (0, 1) \to (0, 1) \]

be the function defined by

\[ g(t) = \frac{27k_0^3}{8} (1 + r_0)^{t} \quad \forall t \in (0, 1). \]

Further let \( \| \hat{x} - x_{n, \alpha} \| \leq \rho, \) with \( \rho = \frac{1}{M} (1 - \frac{3}{2} + M) \frac{\delta + \epsilon_{\delta}}{\sqrt{\alpha}} \) and

\[ \gamma_{n} := M \rho + \frac{3}{2} M (\frac{\delta + \epsilon_{\delta}}{\sqrt{\alpha}}). \]

**Theorem 3.2:** Let \( x_{n, \alpha}^{h, \delta} \) and \( g \) be as in equation (11) and (12) respectively, \( x_{n+1, \alpha}^{h, \delta} \) and \( x_{n, \alpha}^{h, \delta} \) as in (9) and (8) respectively with \( \delta \in (0, \delta_i], \alpha = \alpha_i \) and \( \epsilon_{n} \in (0, \epsilon_0] \).

Then the following hold:

- a) \[ \| x_{n+1, \alpha}^{h, \delta} - x_{n-1, \alpha}^{h, \delta} \leq \frac{3k_0^3}{2} \gamma_{n-1, \alpha}^{h, \delta} \]
- b) \[ \| x_{n, \alpha}^{h, \delta} - x_{n-1, \alpha}^{h, \delta} \leq \frac{3k_0^3}{2} \gamma_{n, \alpha}^{h, \delta} \]
- c) \[ \| y_{n, \alpha}^{h, \delta} - y_{n-1, \alpha}^{h, \delta} \leq g(\epsilon_{n-1, \alpha}^{h, \delta}) \| y_{n, \alpha}^{h, \delta} - x_{n-1, \alpha}^{h, \delta} \|; \]
d) \( g(e^h_\alpha) \leq g(y_\rho) \mu^{2}, \quad \forall n \geq 0; \)

e) \( e^h_\alpha \leq g(y_\rho) \mu^{(n-1)/2} \), \( \forall n \geq 0. \)

**Proof.** Proofs of a), b) and c) are analogous to the proof of corresponding results of Theorem 3.4 in [5]. Further, since for \( \mu \in (0,1), \)
\( g(\mu t) \leq \mu^{2} g(t), \quad \forall t \in (0,1), \) by c) we have,
\[ g(e^h_\alpha) \leq g(e^h_\alpha) \mu^{2} \] 
\[ e^h_\alpha \leq g(e^h_\alpha) \mu^{(n-1)/2} \] 
\[ \leq g(e^h_\alpha) \mu^{(n-1)/2} \] 
\[ \leq g(e^h_\alpha) \mu^{(n-1)/2} e^h_{\alpha} \] 

(13)

provided \( e^h_\alpha < 1, \forall n \geq 0. \) From (13) it is clear that, \( e^h_\alpha < 1 \) if \( e^h_\alpha < 1. \) Now since \( g \) is monotonic increasing and \( e^h_\alpha \leq y_\rho, \) (see [5], equation (3.28)) we have \( g(e^h_\alpha) \leq g(y_\rho). \) This proves d) and e).

**Theorem 3.3:** Let \( F(x) = \frac{1}{1-g(y_\rho)} + (1 + r_\alpha) \) and

the assumptions of Theorem 3.2 hold. Then \( \chi^h_\alpha \), \( Y_\alpha \in B_{h}(P_{x_\alpha}), \) for all \( n \geq 0. \)

**Proof.** Analogous to the proof of Theorem 3.5 in [5]. The main result of this section is the following Theorem.

**Theorem 3.4:** Let \( \chi^h_\alpha \) and \( \chi^h_\alpha \) be as in (8) and (9) respectively and assumptions of Theorem 3.3 hold. Then \( \chi^h_\alpha \) is a Cauchy sequence in \( B_{h}(P_{x_\alpha}) \) and converges to \( \chi^h_\alpha \in B_{h}(P_{x_\alpha}). \) Further \( P_{x_\alpha} F(x^h_\alpha) + \frac{\alpha}{c}(x^h_\alpha - x_\alpha) = P_{x_\alpha} x^h_\alpha \) and \( \chi^h_\alpha - \chi^h_\alpha \) is Cauchy.

where
\[ C_0 = \left( \frac{1}{1-g(y_\rho)} + (1 + r_\alpha) \right)^{\frac{3}{2}} \] 
\[ \gamma_1 = -\log g(y_\rho). \]

**Proof.** Analogous to the proof of Theorem 3.6 in [5].

The following Assumption is used in the further analysis.

**Assumption 3.5:** There exists a continuous, strictly monotonically increasing function \( \varphi : (0, b) \to (0, \infty) \) with \( b \geq F(x_\alpha) \) satisfying:

\[ \lim \lambda \to 0 \varphi(\lambda) = 0; \]

\[ \sup \alpha \varphi(\alpha) \leq \varphi(\alpha), \quad \forall \lambda < (0, b] \]

\[ \lambda \geq 0, \] 

There exists \( v \in X \) with \( \|v\| \leq 1 \) (cf. [8]) such that \( x_\alpha - \hat{x} = \varphi(F(x_\alpha))v. \)

**Theorem 3.7** (see [5], Theorem 3.8) Suppose \( x^h_\alpha \) is the solution of (10) and Assumption 2.1 and Theorem 3.6 hold. In addition if \( r_\alpha < 1, \) then
\[ \|x^h_\alpha - x^h_\alpha\| \leq \frac{2}{1-r_\alpha} \frac{\delta + \epsilon_\rho}{\sqrt{\alpha}}. \]

The following Theorem is a consequence of Theorem 3.4, Theorem 3.6 and Theorem 3.7.

**Theorem 3.8:** Let \( x^h_\alpha \) be as in (9), assumptions in Theorem 3.4, Theorem 3.6 and Theorem 3.7 hold. Then
\[ \|x^h_\alpha - x^h_\alpha\| \leq \frac{\|x^h_\alpha - x^h_\alpha\|}{\sqrt{\alpha}} \]

where \( \varphi_0(\alpha) + (2 + \frac{4}{\mu^{2}} - 1) \mu \psi^{-1}(\delta + \epsilon_\rho) \), and \( \gamma_1 \) and \( \alpha \) are as in Theorem 3.4.

**Theorem 3.9:** Let \( x^h_\alpha \) be as in (9) and assumptions in Theorem 3.8 hold. Further let \( \varphi_0(\alpha) \leq \alpha \) and
\[ n_\alpha := \min \{n : e^{-n\gamma} \leq \delta + \epsilon_\rho \}. \]

Then \( \|x^h_\alpha - x^h_\alpha\| \leq O(\psi^{-1}(\delta + \epsilon_\rho)). \)

**IV. Algorithm**

**Note** that for \( i, j \in \{0, 1, 2, \ldots, N\}, \)
\[ z_{\alpha} - z_{\alpha} = (\alpha - \alpha_i)(P_{h}K^*K^* + I)^{-1}(P_{h}K^*K^* + I)P_{h}K^*(f^h_\alpha - KF(x_\alpha)). \]

Therefore the balancing principle algorithm associated with the choice of the parameter specified in Section II involves the following steps.

- Choose \( \alpha \) such that \( \delta \geq \epsilon_\rho \) and \( \mu > 1; \)
- \( \alpha_0 = \mu^{2} \alpha_0; \)
- solve for \( w_i \)
\[ (P_{h}K^*K^* + I)w_i = P_{h}K^*(f^h_\alpha - KF(x_\alpha)); \] (14)
\begin{itemize}
  \item Solve for \( j < i \),
  \[
  z_{ij}^{h,\delta} := (P_n K^* K P_n + \alpha, I) z_{ij}^{h,\delta} = (\alpha_j - \alpha_i) w_i; \quad (15)
  \]
  \item If \( \| z_{ij}^{h,\delta} \| > 4 C (\delta + \varepsilon_\delta) \sqrt{\| \alpha \|} \), then take \( k = i - 1 \);
  \item Otherwise, repeat with \( i + 1 \) in place of \( i \).
  \item Choose \( n_k = \min \{ n : e^{-\gamma n^4} \leq \delta + \varepsilon_\delta \} \)
  \item Solve \( x_{nk, \alpha_k}^{h,\delta} \) using the iteration (9).
\end{itemize}

V. IMPLEMENTATION OF THE METHOD

In this section we present an example for implementing the algorithm mentioned in above section. We take an example (see [7], section 4.3.) satisfying the assumptions made in this paper. We consider the operator \( K^* : L^2(0, 1) \to L^2(0, 1) \) defined by \( K(x(t)) = \int_0^1 k(t, x(s)) ds \)
and \( F: D(F) \subseteq L^2(0, 1) \to L^2(0, 1) \) defined by
\[
F(u) := \int_0^1 k(t, s) u(s) ds,
\]
where
\[
k(t, s) = \begin{cases}
  (1-t) x(t), & 0 \leq s \leq t \leq 1 \\
  (1-s) x(t), & 0 \leq t \leq s \leq 1.
\end{cases}
\]

Then for all \( x(t), y(t) : x(t) > y(t) \):
\[
\langle F(x) - F(y), x - y \rangle = \int_0^1 \int_0^1 k(t, s) (x^3(s) - y^3(s)) ds dt \geq 0.
\]

Thus the operator \( F \) is monotone. The Frechet derivative of \( F \) is given by
\[
F'(u) w = 3 \int_0^1 k(t, s) (u(s))^2 w(s) ds.
\]

So for any \( u \in B(\mathcal{X}, x_0), x_0(s) \geq k_3 > 0, \forall s \in (0, 1) \), we have
\[
F'(u) w = F'(x_0) G(u, x_0) w,
\]
where
\[
G(u, x_0) = \left( \frac{u}{x_0} \right)^2.
\]

Further observe that
\[
[F'(u) - F'(v)] w(s) = 3 \int_0^1 k(t, s) [v^2(s) - u^2(s)] w(s) ds
\]
\[
:= F'(u) \Phi(u, v, w),
\]
where
\[
\Phi(u, v, w) = \left[ \frac{v^2}{u^2} - 1 \right] w.
\]

Thus \( F \) satisfies the Assumption 3.1 (cf. [10], Example 2.7).

Let \( V_n \) be a sequence of finite dimensional subspaces of \( X \) and \( \dim V_n = n + 1 \). We choose the linear splines \( \{ v_1, v_2, \ldots, v_{n+1} \} \) in a uniform grid of \( n + 1 \) points in \([0, 1]\) as a basis of \( V_n \). Let \( P_n = P_n^\perp \) denote the orthogonal projection on \( X \) with range \( R(P_n) = V_n \). We assume that \( \| P_n x - x \| \to 0 \) as \( h \to 0 \) for all \( x \in X \).

Since \( w_i \in V_n \), \( w_i \) can be written as \( \sum_{k=1}^{n+1} \lambda_i v_k \) for some scalars \( \lambda_i, \lambda_2, \ldots, \lambda_{n+1} \) and \( w_i \) is a solution of (14) if and only if \( \mathcal{X} = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})' \) is the unique solution of \( (M_n + \alpha, B_n) \mathcal{X} = \mathcal{A} \)
where \( M_n = (\langle K y, K v_i \rangle), i = 1, 2, \ldots, n + 1 \)
\[
B_n = (\langle v_i, v_j \rangle), i = 1, 2, \ldots, n + 1
\]
and \( \mathcal{A} = (\langle P_n K^* (f - K F(x_0)), v_i \rangle)' \), \( i = 1, 2, \ldots, n + 1 \).

One can see from (15) that \( z_{ij}^{h,\delta} \in V_n \) and hence
\[
z_{ij}^{h,\delta} = \sum_{k=1}^{n+1} \mu_k v_k
\]
for some \( \mu_k, k = 1, 2, \ldots, n + 1 \). Then, for \( j < i \), \( z_{ij}^{h,\delta} \) is a solution of \( (P_n K^* K P_n + \alpha, I) z_{ij}^{h,\delta} = (\alpha_j - \alpha_i) w_i \) if and only if \( \mu_i \) is a solution of (14) if and only if the unique solution of \( (M_n + \alpha, B_n) \mu_i = \mathcal{B} \) where \( \mathcal{B} = (\langle (\alpha_j - \alpha_i) w_j, v_i \rangle) \).

Compute \( z_{ij}^{h,\delta} \) till \( \| z_{ij}^{h,\delta} \| \geq 4 C (\delta + \varepsilon_\delta) \sqrt{\| \alpha \|} \) and fix \( k = i - 1 \). Now choose \( n_k = \min \{ n : e^{-\gamma n^4} \leq \delta + \varepsilon_\delta \} \).

Let \( \xi^n = (\xi_1^n, \xi_2^n, \ldots, \xi_{n+1}^n) \), \( \eta^n = (\eta_1^n, \eta_2^n, \ldots, \eta_{n+1}^n) \),
\[
\gamma_{nk, \alpha_k}^{h,\delta} = \sum_{i=1}^{n+1} \xi_i^n v_i \text{ and } \gamma_{nk, \alpha_k}^{h,\delta} = \sum_{i=1}^{n+1} \eta_i^n v_i.
\]
Then using (8) we get
\[
(P_i F'(\gamma_{nk, \alpha_k}^{h,\delta})) + \frac{\alpha_k}{c} \sum_{i=1}^{n+1} (\xi_i^n - \eta_i^n) v_i
\]
\[
= \sum_{i=1}^{n+1} \lambda_i v_i - \sum_{i=1}^{n+1} P_i F'(\gamma_{nk, \alpha_k}^{h,\delta}) v_i + \frac{\alpha_k}{c} \sum_{i=1}^{n+1} (\xi_i^n - \eta_i^n) v_i,
\]
where \( t_1, t_2, \ldots, t_{n+1} \) are the grid points.

Observe that \( (\gamma_{nk, \alpha_k}^{h,\delta}, -\gamma_{nk, \alpha_k}^{h,\delta}) \) is a solution of (8) if and only if \( (\xi^n - \eta^n)' = (\xi_1^n - \eta_1^n, \xi_2^n - \eta_2^n, \ldots, \xi_{n+1}^n - \eta_{n+1}^n)' \) is the unique solution of \( (Q + \frac{\alpha_k}{c} B_n)(\xi^n - \eta^n) = B_n(\mathcal{X} - F_{h,1} + \frac{\alpha_k}{c} (X_0 - \mathcal{X})) \), where \( Q_n = (F'(\gamma_{nk, \alpha_k}^{h,\delta}), v_i, v_j), i, j = 1, 2, \ldots, n + 1, \)
\[
F_{h,1} = [F'(\gamma_{nk, \alpha_k}^{h,\delta})(t_1), F'(\gamma_{nk, \alpha_k}^{h,\delta})(t_2), \ldots, F'(\gamma_{nk, \alpha_k}^{h,\delta})(t_{n+1})]' \]
\[ X_0 = [x_{0}(t_1), x_{0}(t_2), \ldots, x_{0}(t_{n+1})]^T. \]

Further from (9) it follows that
\[
(PF(x_{n}, \alpha)) + \alpha \frac{\partial}{\partial x}(x_{n}, \alpha) - y_{n, \alpha})
= PF(x_{n+1}, \alpha) + \alpha \frac{\partial}{\partial x}(x_{n+1}, \alpha) - y_{n, \alpha}).
\] (16)

Thus \((x_{n+1, \alpha} - y_{n, \alpha})\) is a solution of (16) if and only if
\[
(\eta^{(n+1)} - \xi) = (\eta^{(n+1)} - \xi, \eta^{(n+1)} - \xi, \ldots, \eta^{(n+1)} - \xi)^T
\]
is the unique solution of
\[
(Q_{n} + \alpha \frac{\partial}{\partial x}B_{n})(\eta^{(n+1)} - \xi) = B_{n}X_{0} - F + \alpha \frac{\partial}{\partial x}(X_{0} - \xi),
\]
where \(Q_{n} = \langle F'(y_{n, \alpha})v_{i}, v_{j}\rangle, i, j = 1, 2, \ldots, n+1, \)
\[
F_{h2} = [F(x_{n, \alpha})(t_{1}), F(x_{n, \alpha})(t_{2}), \ldots, F(x_{n, \alpha})(t_{n+1})]^T.
\]

To illustrate the above method in our computation, we take \(f(t) = \frac{1}{110}(t_{13} - t_{15} + \frac{25}{156}t_{15})\) and \(f' = f + \delta.\) Then the exact solution \(\hat{x}(t) = t^{3}\). We use \(x_{0}(t) = t^{3} + \frac{3}{56}(t - t_{6})\) as our initial guess, so that the function \(x_{0} - \hat{x}\) satisfies the source condition \(x_{0} - \hat{x} = \phi_{1}(F'(x_{0}))\|\) where \(\phi_{1}(\lambda) = \lambda.\) Thus we expect to have an accuracy of order at least \(O((\delta + \epsilon_{k})^2).\)

We choose \(\alpha_{n} = (1.3)^{n}+\epsilon_{k}+\mu, \mu = 1.3.\)
\[
\delta + \epsilon_{k} = 0.0667, \gamma_{\rho} = 0.8173, \text{ and } g(\gamma_{\rho}) = 0.54
\]
approximately. For all \(n\) the number of iteration \(n_{x} = 1.\) The results of the computation are presented in Table I. The plots of the exact and the approximate solutions obtained are given in Fig. 1.

**TABLE I: ITERATIONS AND CORRESPONDING ERROR ESTIMATES**

<table>
<thead>
<tr>
<th>(n)</th>
<th>(k)</th>
<th>(\delta + \epsilon_{k})</th>
<th>(\alpha)</th>
<th>(|x_{k} - \hat{x}|)</th>
<th>(|x_{k} - \hat{x}| / (\delta + \epsilon_{k})^{2})</th>
</tr>
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<td>8</td>
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<td>0.0682</td>
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**Fig. 1. Curves of the Exact and Approximate solutions.**

**VI. CONCLUSION**

A finite dimensional realization of a Two Step Newton-Tikhonov Projection Method is considered for obtaining an approximate solution of a nonlinear ill-posed Hammerstein type operator equation \(KF(x) = f.\) Under the assumption that the available data is \(f \in L^{2}\) and the nonlinear operator \(F\) is monotone but Frechet derivative of \(F\) is non-invertible, we obtained the convergence of the method. The derived error estimates using an a priori and adaptive method of Pereverzev and Schock (2005) are of optimal order with respect to a general source condition. Numerical Example presented proves the reliability of our method.

**REFERENCES**

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