A New Discontinuous Galerkin Method to Solve Highly Sensitive Troesch's Problem

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***Abstract*—In this paper, we propose a new discontinuous Galerkin finite element (DG) method to solve Troesch's problem, which is highly sensitive for large values of the parameter. This twopoint boundary value problem has been heavily studied since 1960, however, only a few papers have provided a reliable solution for high sensitivity. Therefore, we developed the DG method which has been proved its efficiency for many decades to be a new numerical solver. We demonstrate through computational results compared with those computed by other methods, that the discontinuous Galerkin method provides a quite efficient, accurate and reliable solution. Thus, the DG method is an attractive and competitive alternative to other numerical and semi-analytical techniques to solve highly sensitive nonlinear problems.**

***Index Terms*—Troesch's problem, discontinuous galerkin method, nonlinear boundary value problem.**

I. Introduction

Troesch's problem, which arises in the investigation of the confinement of a plasma column by radiation pressure, was initially introduced and formulated by Weibel [1] and Troesch [2]. Troesch's problem is defined by

$$u'' = \lambda \sinh(\lambda u) \qquad (1)$$

subject to

$$u(0) = 0, \quad u(1) = 1 \qquad (2)$$

Roberts and Shipman [3] combined the multipoint, continuation and perturbation methods to provide an accurate solution of the problem for $\lambda \leq 5$. Jones [4], Troesch [5], Scott and Watts [6] and Kubicek [7] used the shooting method for solving the problem. Chiou [8] applied a non-iterative method known as method of transformation groups to solve the problem which appeared by that time to be simple and less-time consuming compared to older techniques. Scott [9] presented the invariant embedded method and Scott and Watts [10]-[11] presented a combined superposition and quasi-linearization procedure as other alternatives to solve the problem. In 1976, an outstanding paper by Roberts and Shipman [12] appeared providing a closed form solution to the problem in terms of Jacobian elliptic functions. Snyman [13] implemented the inverse shooting method and has been successful in solving this problem for large value of λ. Snyman's, Roberts' and Scott's results turned to be considered as benchmark solution for recent studies.

In our paper, we adopt the discontinuous Galerkin (DG) method to solve the Troesch's problem. This DG method can overcome the difficulty of the problem in term of high slope and concavity for large λ near x=1. Unlike the standard finite element method, the DG variational (primal or mixed) formulation involves some jump terms due to the choice of the solution and/or its derivative (and/or higher-order derivative) induced from the integration by parts on the boundary elements. In fact, researchers use the term numerical flux to define the solution on the boundaries. The choice of the numerical flux is the most delicate and crucial aspect of the definition of the DG method as it affects its stability and accuracy, as well as properties such as sparsity and symmetry of the stiffness matrix.

The paper is organized as follows: In §II we present the discontinuous Galerkin method applied to the Troesch's problem. In III we show several numerical results for different values of λ and we conclude with a few remarks in IV.

II. The Discontinuous Galerkin Method

In this section, the discontinuous Galerkin finite element method is developed and implemented for solving the Troesch's problem defined by

$$u'' = \lambda \sinh(\lambda u) \qquad (3)$$

subject to

$$u(0) = 0, \quad u(1) = 1 \qquad (4)$$

where λ is a positive constant.

In order to implement the discontinuous Galerkin (DG) method, we first create a partition, $x_k = k\Delta x$, k=0, 1, 2, ..., N+1, $\Delta x = \frac{1}{N+1}$ with and define the piecewise polynomial spaces

$$S^{n,p} = \{U : U|_{I_k} \in P_p\} \qquad (5)$$

where P_p denotes the space of Legendre polynomials of degree p which will be adopted as basis functions.

We define the weak discontinuous Galerkin (DG) formulation for (3) by multiplying it by a test function, and then integrating over I_k. After integrating by parts, we obtain

$$u'v\big|_{x_k}^{x_{k+1}} - uv'\big|_{x_k}^{x_{k+1}} + \int_{x_k}^{x_{k+1}} uv''dx - \int_{x_k}^{x_{k+1}} \lambda \sinh(\lambda u)dx = 0 \qquad (6)$$

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Let us replace u by $U_k(x) = U|_{I_k} \in P_p$ and v by $V \in P_p$ in (6), we obtain for k=0, 1, 2, ⋯, N and $\forall V \in P_p$

$$\widehat{U}_k'(x_{k+1})V(x_{k+1}^-) - \widehat{U}_k'(x_k)V(x_k^+) - \widehat{U}_k(x_{k+1})V'(x_{k+1}^-) + \widehat{U}_k(x_k)V'(x_k^+) + \int_{x_k}^{x_{k+1}} U_k V'' dx - \int_{x_k}^{x_{k+1}} \lambda \sinh(\lambda U_k) dx = 0 \quad (7)$$

where $\widehat{U}_k(x_k)$, $\widehat{U}_k(x_{k+1})$, $\widehat{U}_k'(x_k)$ and $\widehat{U}_k'(x_{k+1})$ are called numerical fluxes. These terms arise from a double integration by parts and an appropriate choice of these fluxes will define a stable DG method. Therefore, let us choose for k=1, 2, ⋯, N−1

$$\widehat{U}_k(x_{k+1}) = U_k(x_{k+1}^-), \quad \widehat{U}_k(x_k) = U_{k-1}(x_k^-)$$
$$\widehat{U}_k'(x_{k+1}) = U_{k+1}'(x_{k+1}^+), \quad \widehat{U}_k'(x_k) = U_k'(x_k^+)$$

for k=0

$$\widehat{U}_0(x_1) = U_0(x_1^-), \quad \widehat{U}_0(x_0) = 0$$

and

$$\widehat{U}_0'(x_1) = U_1'(x_1^+), \quad \widehat{U}_0'(x_0) = U_0'(x_0^+)$$

for k=N

$$\widehat{U}_N(1) = 1, \quad \widehat{U}_N(x_N) = U_{N-1}(x_N^-)$$
$$\widehat{U}_N'(1) = U_N'(1^-), \quad \widehat{U}_N'(x_N) = U_N'(x_N^+)$$

Therefore, the discrete formulation consists of determining $U_k(x) = U|_{I_k} \in P_p$, such that $\forall V \in P_p$

$$U_1'(x_1^+)V(x_1^-) - U_0'(0^+)V(0^+) - U_0(x_1^-)V'(x_1^-) + \int_0^{x_1} U_0 V'' dx - \int_0^{x_1} \lambda \sinh(\lambda U_0) dx = 0 \quad (8)$$

for k=1, 2, ⋯, N−1

$$U_{k+1}'(x_{k+1}^+)V(x_{k+1}^-) - U_k'(x_k^+)V(x_k^+) - U_k(x_{k+1}^-)V'(x_{k+1}^-) + U_{k-1}(x_k^-)V'(x_k^+) + \int_{x_k}^{x_{k+1}} U_k V'' dx - \int_{x_k}^{x_{k+1}} \lambda \sinh(\lambda U_k) dx = 0 \quad (9)$$

and

$$U_N'(1^-)V(1^-) - U_N'(x_N^+)V(x_N^+) + U_{N-1}(x_N^-)V'(x_N^+) + \int_{x_N}^1 U_N V'' dx - \int_{x_N}^1 \lambda \sinh(\lambda U_N) dx = V'(1^-) \quad (10)$$

We note that the DG solutions on each element I_k can be written for k=0, 1, ⋯, N as

$$U_k(x) = \sum_{i=0}^p c_{i,k}\psi_i(x) = \sum_{i=0}^p c_{i,k}\widehat{\psi}_i(\xi) \quad (11)$$

$$U_k'(x) = \sum_{i=0}^p c_{i,k}\psi_i'(x) = \sum_{i=0}^p \frac{2}{\Delta x} c_{i,k}\widehat{\psi}_i'(\xi) \quad (12)$$

where $\psi_i(x)$ are Legendre polynomial of degree i on the interval I_k and $\widehat{\psi}_i(\xi)$ are the mapped Legendre polynomial of degree i to the standard interval [−1,1]. Then, we choose the test function V to be $\widehat{\psi}_j$ and substitute (11) and (12) in (8)-(10) to obtain for j=0,1 , 2, ⋯, p

$$\sum_{i=0}^p \frac{2}{\Delta x}\widehat{\psi}_i'(-1)\widehat{\psi}_j(1)c_{i,1} - \sum_{i=0}^p \frac{2}{\Delta x}\widehat{\psi}_i'(-1)\widehat{\psi}_j(-1)c_{i,0} - \sum_{i=0}^p \frac{2}{\Delta x}\widehat{\psi}_i(1)\widehat{\psi}_j'(1)c_{i,0} + \sum_{i=0}^p \left(\frac{2}{\Delta x}\right)^2 \left(\int_{-1}^1 \widehat{\psi}_i(\xi)\widehat{\psi}_j''(\xi)d\xi\right)c_{i,0} - \int_{-1}^1 \frac{\Delta x}{2}\lambda \sinh\left(\lambda \sum_{i=0}^p c_{i,0}\widehat{\psi}_i(\xi)\right)\widehat{\psi}_j(\xi)d\xi = 0 \quad (13)$$

for k=1, 2, ⋯, N−1

$$\sum_{i=0}^p \frac{2}{\Delta x}\widehat{\psi}_i'(-1)\widehat{\psi}_j(1)c_{i,k+1} = -\sum_{i=0}^p \frac{2}{\Delta x}\widehat{\psi}_i'(-1)\widehat{\psi}_j(-1)c_{i,k} - \sum_{i=0}^p \frac{2}{\Delta x}\widehat{\psi}_i(1)\widehat{\psi}_j'(1)c_{i,k} + \sum_{i=0}^p \left(\frac{2}{\Delta x}\right)^2 \left(\int_{-1}^1 \widehat{\psi}_i(\xi)\widehat{\psi}_j''(\xi)d\xi\right)c_{i,k} - \int_{-1}^1 \frac{\Delta x}{2}\lambda \sinh\left(\lambda \sum_{i=0}^p c_{i,k}\widehat{\psi}_i(\xi)\right)\widehat{\psi}_j(\xi)d\xi = 0 \quad (14)$$

for k=N

$$\sum_{i=0}^p \frac{2}{\Delta x}\widehat{\psi}_i'(1)\widehat{\psi}_j(1)c_{i,N} - \sum_{i=0}^p \frac{2}{\Delta x}\widehat{\psi}_i'(-1)\widehat{\psi}_j(-1)c_{i,N} + \sum_{i=0}^p \frac{2}{\Delta x}\widehat{\psi}_i(1)\widehat{\psi}_j'(-1)c_{i,N-1} + \sum_{i=0}^p \left(\frac{2}{\Delta x}\right)^2 \left(\int_{-1}^1 \widehat{\psi}_i(\xi)\widehat{\psi}_j''(\xi)d\xi\right)c_{i,N} - \int_{-1}^1 \frac{\Delta x}{2}\lambda \sinh\left(\lambda \sum_{i=0}^p c_{i,N}\widehat{\psi}_i(\xi)\right)\widehat{\psi}_j(\xi)d\xi = \widehat{\psi}_j^1(1) \quad (15)$$

III. Numerical Experiment

Let us now implement the DG method to solve Troesch's problem for different values of the parameter λ. The challenge throughout the years for this nonlinear parametric problem consists of finding the solution for large values of λ. Therefore, in this study, we provide the DG solution of the Troesch's problem for λ=1,3,5,10,15,20 and we show the efficiency of the method compared to existing numerical and semi-analytical results [5], [9], [12].

We solve (3) using a uniform mesh with Δx=0.01 and p=2λ for sake of accuracy mainly for large values of λ where an approximation by polynomials of higher degree is required. We plot the discontinuous Galerkin solution *U* versus *x* in Fig. 1 for different values of the parameter λ.

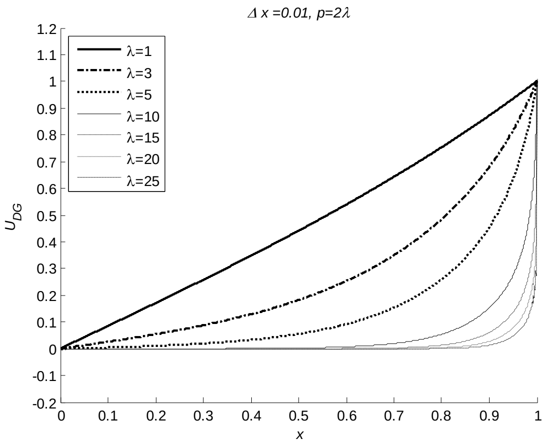

Fig. 1. The DG solution of the troesch's problem for various λ.

Table I exhibits the pointwise DG solution at different values of *x* and is compared to the benchmark solution provided in [9] for λ=10 along with the error terms. These computational results reveal the accuracy and the efficiency of the DG method and unveiled to be a reliable solver for this kind of parametric nonlinear problem.

Moreover, in Table II, we presented an accurate pointwise DG solution for large values of λ which could be taken as a reference solution for future studies.

TABLE I: THE DG SOLUTION OF THE TROESCH'S PROBLEM FOR λ=10

x	$U(x)$[9]	$U_{DG}(x)$	$U(x)[9]-U_{DG}(x)$
0	0	0	0
0.1	0.0000421118367	0.0000421118992	-0.0062500000000(-08)
0.2	0.0001299639238	0.0001299641158	-0.0192000000018(-08)
0.3	0.0003589778855	0.0003589784013	-0.0515799999988(-08)
0.4	0.0009779014227	0.0009779027718	-0.1349100000117(-08)
0.5	0.0026590171780	0.0026590204903	-0.3312300000252(-08)
0.6	0.0072289246952	0.0072289312128	-0.6517599999630(-08)
0.7	0.0196640602566	0.0196640630970	-0.2840400002007(-08)
0.8	0.0537303295856	0.0537303293505	0.0235100001966(-08)
0.9	0.1521140787863	0.1521140764047	0.2381600000544(-08)
1	1	1.0000000090671	-0.9067099959736(-08)

TABLE II: THE DG SOLUTION OF THE TROESCH'S PROBLEM FOR λ=15,20,25

x	$U_{DG}(x)$,λ=15	$U_{DG}(x)$,λ=20	$U_{DG}(x)$,λ=25
0	0	0	0
0.1	0.000000347003229	0.000000002989864	0.000000000026818
0.2	0.000001632587467	0.000000022496907	0.000000000328906
0.3	0.000007334025683	0.000000166285667	0.000000004007079
0.4	0.000032872677760	0.000001228701537	0.000000048816234
0.5	0.000147325995007	0.000009078945592	0.000000594703476
0.6	0.000660270775636	0.000067084840902	0.000007244971505
0.7	0.002959243782406	0.000495694649225	0.000088261830496
0.8	0.013272815796087	0.003663117627065	0.001075265295340
0.9	0.060450171661991	0.027230987802378	0.013128600791286
1	1.000128195252954	1.000893657815884	1.007705271805717

{TC "2 The DG solution of the Troesch's problem for λ=15,20,25." \ft}

IV. CONCLUSION

An efficient and accurate discontinuous Galerkin method has been developed to solve the Troesch's problem for different values of λ. This DG computational scheme was successful to provide a reliable solution and to avoid the difficulty of the problem caused by large values of λ. This success was illustrated through the computational results shown earlier which reveal an outstanding agreement with the benchmark solutions. Therefore, due to its computational simplicity and efficiency, the DG method for solving Troesch's problem with a wide range of λ could be considered for future work as a reference trustworthy solver.

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