Robustness of Tukey's Control Chart in Detecting a Changes of Parameter of Skew Distributions

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Abstract—**The observations are not always identically and independently distributed (i.i.d.) in continuous manufacturing which most observations is skewed away normal distribution. In this research aim to study the performance of Tukey's control chart for detecting a change in parameter when observation are from skew distributions such as exponential and Laplace distributions. Also, the performance of Tukey's control chart is compared with Shewhart and Exponentially Weighted Moving Average (EWMA) charts. The Average Run Length is commonly measured the performance of control chart. The ARL0 is usually used when the process is in-control (sufficient large) and the ARL1 is correctly signaled to be out-of-control (minimum). The Tukey's control chart is superior to other charts for large shift; however, the EWMA performs better for small to moderate shifts. The Tukey's control chart is very easy to setup the control limits and use a simple statistical concept.**

Index Terms—**Average Run Length, Skew Distributions, EWMA, Tukey's Control Chart, Robustness.**

I. INTRODUCTION

Statistical Process Control (SPC) charts are widely used for monitoring, measuring, controlling and improving quality of production in many areas of application, for example, in industry and manufacturing, finance and economics, epidemiology and health care, environmental sciences and other fields. Control charts are usually designed and evaluated under the assumption that the observations from a process are independent and identically distributed (i.i.d.) and from a normal distribution. In real applications, there are many situations in which the process data come from a non-normal distribution, for example, an Exponential, Laplace, Student-t or Gamma distribution (see, e.g., Borror et al. [1]; Stoumbos and Reynolds [2]; Mititelu et al. [3]). Processes with data from a non-normal distribution need to be monitored by appropriate control charts.

Recently, many types of control charts are proposed which an appropriate control chart must be selected under many assumptions and several factors. Specially, for this kind of process monitoring, individual control charts take only one sample to measure due to economic issue for a company. Consequently, Tukey's control chart has been popular used for individual process which Alemi (2004) who was first proposed. Torng and Lee (2008) [4] and Torng et al. (2009) [5] have been investigated the average

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run length of Tukey's control chart. There are many advantage of using Tukey's control chart which it is easy to use and simple control limits setup. It can be used with not only non-Normal observations but also when distribution of process is unknown. Furthermore, Tukey's control chart does not sensitive to unusual data such as an outlier.

Consequently, this paper aim to study the performance of Tukey's control chart robust to the skew distribution processes such exponential and Laplace distributions. They are usually represented as lifetime of products and growth rate of a company, respectively.

II. CONTROL CHARTS AND PROPERTIES

Let $X_t, t = 1, 2, \dots$ be a sequence of independent identically distributed (i.i.d.) random variables with a distribution $F(x, \alpha)$, where α is a parameter. It is usually assumed that under some "standard" (or "in-control") conditions the parameter has a known in-control value $\alpha = \alpha_0$. Then at some unknown time v, which is called the change-point time $(v < \infty)$, the parameter α could be changed to an "out-of-control" value $\alpha \neq \alpha_0$.

A. Tukey's Control Chart with Symmetrical Control Limits

In 2004, Alemi [6] who first presented the Tukey's control chart which applied the principle of Box plot to obtain the control limits. The control limits of Tukey's control chart are proposed by Torng and Lee (2008) as follows:

$$UCL = F^{-1}(0.75) + L \times IQR$$

$$\text{and} \quad LCL = F^{-1}(0.25) - L \times IQR \quad (1)$$

where UCL and LCL are upper and lower control limits, respectively. $F^{-1}(0.75)$ and $F^{-1}(0.25)$ are the third quartile (Q_3) and first quartile (Q_1) and IQR is the Inter-Quartile Range (IQR = Q_3-Q_1). The value of L is a coefficient of control limit which this value of L is usually set as 1.5 for the case of a normal distribution assumption (Ryan (2000)).

B. Exponentially Weighted Moving Average (EWMA) control chart.

The Exponentially Weighted Moving Average (EWMA) statistics for the discrete time case has the following form

$$Z_t = \lambda(X_t - \mu_0) + (1 - \lambda)Z_{t-1}, \qquad 0 < \lambda < 1, \quad (2)$$

where λ is a weighting factor for previous observations.

The target in-control parameter α_0 is supposed to be steady and the initial value Z_0 is usually chosen to be the process in-control parameter, i.e. $Z_0 = \alpha_0$. If the observations X_t are independent random variables with variance σ^2, then the variance of EWMA statistics Z_t is

$$\sigma_{X_t}^2 = \sigma^2 \left(\frac{\lambda}{2-\lambda}\right)[1-(1-\lambda)^{2t}], \ t=1, 2, \ldots \quad (3)$$

Since $0 < 1 - \lambda < 1$, we have that $(1-\lambda)^{2t} \to 0$ as $t \to \infty$, and therefore the asymptotic value of the variance is

$$\sigma_{X_t}^2 = \sigma^2 \left(\frac{\lambda}{2-\lambda}\right). \quad (4)$$

The standard deviation used in the limits is usually the asymptotic value. Using the expression in Equation (3), the control limits of the EWMA chart is the following:

$$UCL = H_U = \alpha_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}}$$

and $LCL = H_L = \alpha_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \quad (5)$

where L is a constant to be chosen. The process will be declared to be in an out-of-control state when $X_t > H_U$ or $X_t < H_L$ (see Sukparungsee and Novikov [7])

C. Shewhart Control Chart

Let $X_1, X_2, \ldots, X_t$ be independent and identically random variables (observations).

The level A and -A are called the upper control limit (UCL) and the lower control limit (LCL). Traditionally,

$$UCL = A = \alpha_0 + L\frac{\sigma}{\sqrt{n}}$$

and $LCL = -A = \alpha_0 - L\frac{\sigma}{\sqrt{n}}, \quad (6)$

where L is constant (usually L=3 for the case of normal distribution)

As stated above, one of the popularly important characteristics for SPC charts is Average Run Length (ARL) – the expectation of an alarm time (τ) is taken to signal (wrongly) about a possible change. Ideally, an acceptable ARL of in-control process should be enough large and a small ARL when the process is out-of-control, so-called Average of Delay time (AD) - the expectation of delay for true alarm time.

In the case of Tukey's and Shewhart control charts, ARL and AD can be easily calculated in terms of the error probabilities- the probability of type I error (P_I) (the test rejects "true" H_0:signal occurs when a point falls outside the control limit when there is no real change in parameter) and the probability of type II error (P_{II}) (the test rejects "true" H_1: no signal occurs when a point falls outside the control limit when there is a real change in parameter). The corresponding formulas are

$$ARL_0 = \frac{1}{P_I}$$

and

$$ARL_1 = \frac{1}{1-P_{II}}, \quad (7)$$

where $P_I = 1 - \int_{LCL}^{UCL} f(X)dx$ and

$P_{II} = 1 - \int_{LCL-\delta\sigma}^{UCL-\delta\sigma} f(X)dx$, $f(x)$ be the probability density function (pdf) of population, and δ is the shift size coefficient; $\delta = \alpha - \alpha_0 / \sigma$.

III. COMPARISON AND NUMERICAL RESULTS

The selected skew population in this study is exponential and laplace (double exponential) distributions which in-control parameter of exponential is 1, 1/3 and 1/5 and shift size (δ) is 0.01, 0.05, 0.1, 0.5, 1, 1.5, 2, 3, 4, 5. For the latter, in-control parameter is Laplace(2,1) and shift size (δ) is 0.1, 0.2, 0.3, 0.5, 1, 2, 3. The performance of Tukey's control chart is compared with Shewhart and EWMA charts when given ARL = 100 and 370.4. Table I and II show the comparison of the performance of control charts when observations are exponential and Laplace distributed, respectively.

TABLE I: COMPARISON OF THE PERFORMANCE OF CONTROL CHARTS FOR THE CASE OF EXPONENTIAL DISTRIBUTION

α_0	δ	$ARL_0 = 100$			$ARL_0 = 370.4$		
		Tukey	Shewhart	EWMA $\lambda = 0.05$	Tukey	Shewhart	EWMA $\lambda = 0.05$
1	0.05	95.914	79.551	70.064[*]	129.571	277.764	214.345[*]
	0.1	91.267	64.937	51.053[*]	123.183	216.993	140.147[*]
	0.5	61.194	21.660	13.699[*]	82.598	51.053	24.521[*]
	1	37.107	10.003	7.112[*]	50.105	18.929	11.299[*]

	2	13.648	4.593	3.964*	18.431	7.110	5.767*
	3	5.021	3.128	2.949*	6.780	4.353	4.068*
	4	1.847*	2.494	2.490	2.494*	3.243	3.246
	5	1*	2.175	2.150	1*	2.696	2.805
1/3	0.05	95.465	92.066	90.617*	352.187	333.407	310.758*
	0.1	90.854	84.956	83.946*	335.025	308.628	263.521*
	0.5	60.907	50.511	49.379*	224.667	159.269	105.053*
	1	36.941	34.280	33.144*	136.146	85.559	56.342*
	2	13.587	15.612	20.462*	50.065	35.005	29.092*
	3	4.999*	9.983	14.817	18.432*	19.325	20.120
	4	1.839*	7.155	11.849	6.774*	12.733	15.720
	5	1*	5.638	9.974	2.493*	9.132	12.881
1/5	0.05	94.917*	95.197	95.541	352.572	348.666	334.028*
	0.1	90.278*	91.378	90.683	334.048	329.791	298.463*
	0.5	60.508*	65.834	64.234	224.709	220.020	157.628*
	1	36.713*	45.451	47.130	136.325	138.821	91.412*
	2	13.646*	27.003	31.419	50.144	68.402	49.127*
	3	4.965*	17.678	23.750	18.446*	40.355	34.034
	4	1.827*	12.760	19.235	6.785*	26.567	26.505
	5	1*	10.065	16.374	2.497*	19.586	21.524

* give minimum ARL_1

TABLE II: COMPARISON OF THE PERFORMANCE OF CONTROL CHARTS FOR THE CASE OF LAPLACE DISTRIBUTION

α_0	δ	ARL$_0$ = 100			ARL$_0$ = 370.4		
		Tukey	Shewhart	EWMA $\lambda = 0.05$	Tukey	Shewhart	EWMA $\lambda = 0.05$
	0.1	86.626	99.372	69.146*	322.806	366.560	169.26*
	0.3	65.899	91.972	43.45*1	242.509	339.133	70.773*
	0.5	49.346	79.620	32.432*	185.115	293.492	45.592*
	1.0	24.127	46.085	20.650*	90.826	169.733	25.679*
	2.0	5.998*	11.824	12.342	21.994	43.618	14.522*
	3.0	1.555*	2.884	8.941	5.433*	10.633	10.321
	4.0	1*	1.213	7.075	1.481*	2.590	8.104

IV. CONCLUSION

We have compared the performance of Tukey's control chart with Shewhart and EWMA charts when observations are from exponential and Laplace distributions. The ARL of Tukey's chart is calculated by the Equation (4) while Shewhart and EWMA approximate the ARL by Monte Carlo simulation. We found that the performance of EWMA is superior to other charts for small to moderate shifts when in-control parameter of exponential distribution is 1 and 1/3 and Tukey's control chart is good to detect large shifts for both case of ARL$_0$=100 and 370.4. When in-control parameter is 1/5 the performance of Tukey's control chart shows the best performance for all shift sizes with both case of ARL$_0$=100 and 370.4. Tukey's control chart is also robust to Laplace distribution process and the numerical results are good agreement with the former case study.

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