Coupled Fixed Point Theorems for a Pair of Weakly Compatible Maps Along with (CLRg) Property in Fuzzy Metric Spaces

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***Abstract*—In this paper, we introduce the notions of E.A. property and (CLRg) property for coupled mappings and generalize the result of Xin-Qi Hu [4] and many others using these notions. Examples supporting our results have also been cited.Our results do not exploit the completeness of the whole space or any of its range space and continuity of maps.**

***Index Terms*—Coupled fixed point, Weakly compatible maps, E.A. property, (CLRg) property.**

I. Introduction

Recently, Bhaskar and Lakshmikantham [2] introduced the concepts of coupled fixed points and mixed monotone property and illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Later on these results were extended and generalized by Sedghi et al. [6] , Fang [3] and Xin-Qi Hu [4] etc.

In 2008, Aamri and Moutawakil [1] introduced the concept of E.A. property in metric space. Recently, Sintunavarat and Kuman [7] introduced a new concept of (CLRg) .The importance of CLRg property ensures that one does not require the closeness of range subspaces.

The intent of this paper is to establish the concept of E.A. property and (CLRg) property for coupled mappings and prove a generalization of the result of Xin-Qi Hu [4].

II. Definitions and Preliminaries

Definition 2.1[8]. A binary operation*: [0, 1] × [0, 1]→ [0, 1] is a continuous t-norm if ([0, 1], *) is a topological abelian monoid with unit 1 s.t. a * b ≤ c * d whenever a ≤ c and b ≤ d , ∀ a, b, c, d ∈ [0,1].

Definition 2.2[4]. Let $\sup_{0<t<1}$ Δ(t, t) = 1. A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at t = 1, where

$\Delta^1(t)$ = t, $\Delta^{m+1}(t)$ = t Δ ($\Delta^m(t)$), m= 1, 2,….., t ∈ [0, 1]. A t-norm Δ is a H-type t-norm iff for any $\lambda \in (0, 1)$, there exists $\delta(\lambda) \in (0, 1)$ such that $\Delta^m(t) > (1-\lambda)$ for all m ∈ N, when t > (1-δ).

The t-norm Δ_M = min. is an example of t-norm of H-type.

Definition 2.3[8].The 3-tuple (*X*, *M*, *) is called a fuzzy metric space if *X* is an arbitrary set, * is a continuous t-norm and *M* is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions:

(FM-1) $M(x, y, 0) > 0$,

(FM-2) $M(x, y, t) = 1$ iff $x=y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) \times M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, t)$: (0,∞) → [0,1] is continuous, for all *x*, *y*, *z* ∈ *X* and *s*, *t* > 0.

(FM-6) $\lim_{n\to\infty} M(x, y, t) = 1$, ∀ *x*, *y* ∈ *X* and *t* > 0.

Definition 2.4[4]. Define $\boldsymbol{\Phi} = \{ \phi : R^+ \to R^+\}$, where $R^+ = [0, + \infty)$ and each $\phi \in \boldsymbol{\Phi}$ satisfies the following conditions:

(ϕ-1) ϕ is non-decreasing;

(ϕ-2) ϕ is upper semicontinuous from the right;

(ϕ-3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi(\phi^n(t))$, $n \in N$.

Clearly, if $\phi \in \boldsymbol{\Phi}$, then ϕ(t) < t for all *t* > 0.

Definition 2.5[5]. An element (*x*, *y*) ∈ *X* × *X* is called a

(i) coupled fixed point of the mapping f: *X* × *X* → *X* if f(*x*, *y*) = *x* , *f*(*y*, *x*) = *y*.

(ii) coupled coincidence point of the mappings f: *X* × *X* → *X* and g: *X* → *X* if

$$f(x, y) = g(x), \quad f(y, x) = g(y).$$

(iii) common coupled fixed point of the mappings f: *X* × *X* → *X* and g: *X* → *X* if

$$x = f(x, y) = g(x), \quad y = f(y, x) = g(y).$$

Definition 2.6[4]. An element *x* ∈ *X* is called a common fixed point of the mappings *f*: *X* × *X* → *X* and *g*: *X* → *X* if *x* = *f*(*x*, *x*) = *g*(*x*).

Definition 2.7[4]. The mappings *f*: *X* × *X* → *X* and g: *X* → *X* are called

(i) commutative if *gf*(*x*, *y*) = *f*(*gx*, *gy*), *gf*(*y*, *x*) = *f*(*gy*, *gx*) for all *x*, *y* ∈ *X*.

(ii) compatible if

$$\lim_{n\to\infty} M(gf(x_n, y_n), f(g(x_n), g(y_n)), t) = 1,$$

$\lim_{n\to\infty} M(gf(y_n, x_n), f(g(y_n), g(x_n)), t) = 1$, for all t > 0 whenever $\{x_n\}$ and $\{y_n\}$ are sequences in *X*, such that $\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$, for *x*, *y* ∈ *X*.

Definition 2.8. The mappings f: *X* × *X* → *X* and g: *X* → *X* are called weakly compatible maps if *f*(*x*, *y*) = *g*(*x*), *f*(*y*, *x*) = *g*(*y*) implies *gf*(*x*, *y*) = *f*(*gx*, *gy*), *gf*(*y*, *x*) = *f*(*gy*, *gx*), for all *x*, *y* in *X*.

Now, we fuzzify the newly defined concepts of E.A Property introduced by Aamri and Moutawakil [1] and (CLRg) property given by Sintunavarat and Kuman [7] for coupled maps as follows

Definition 2.9, Let (*X*, *M*, *) be a FM space. Two maps f: *X*

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$\times X \to X$ and g: $X \to X$ are said to satisfy E.A. property if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty}(f(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and

$\lim_{n\to\infty}(f(y_n, x_n) = \lim_{n\to\infty} gy_n = y$, for some x, y in X.

Definition 2.10. Let $(X, M, *)$ be a FM space .Two maps f: $X \times X \to X$ and g: $X \to X$ are said to satisfy (CLRg) property if there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} M(f(x_n, y_n), gx_n, t) = g(p)$ and

$\lim_{n\to\infty} M(f(y_n, x_n), gy_n, t) = g(q)$, for some p, q in X.

Example 2.1. Let $(X, M, *)$ be a fuzzy metric space, * being a continuous norm with $X = [0, 1]$. Define $M(x, y, t) = \frac{t}{t+|x-y|}$ for all x, y in X and $t > 0$. Also define the maps f: $X \times X \to X$ and g: $X \to X$ by $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$ and $g(x) = \frac{x}{2}$ respectively. Note that 0 is the points of coincidence of f and g. It is clear that the pair (f, g) is weakly compatible on X. We next show that the pair (f, g) is not compatible.

Consider the sequences $\{x_n\} = \{\frac{1}{2} + \frac{1}{n}\}$ and $\{y_n\} = \{\frac{1}{2} - \frac{1}{n}\}$, then

$$f(x_n, y_n) = \frac{1}{4} + \frac{1}{n^2} = f(y_n, x_n), g(x_n) = \frac{1}{4} + \frac{1}{2n}, g(y_n) = \frac{1}{4} - \frac{1}{2n}$$

$$\lim_{n\to\infty} M(f(x_n, y_n), g(x_n), t) = \left[\frac{t}{t + \left|\frac{1}{n^2} - \frac{1}{2n}\right|}\right] \to 1 \neq g(p)$$

for any p.

$$\lim_{n\to\infty} M(f(y_n, x_n), g(y_n), t) = \left[\frac{t}{t + \left|\frac{1}{n^2} - \frac{1}{2n}\right|}\right] \to 1 \neq g(q)$$

for any q.

$M(f(gx_n, gy_n), gf(x_n, y_n), t) = \frac{t}{t+|f(gx_n, gy_n) - gf(x_n, y_n)|} = \frac{t}{t + \frac{1}{4}(\frac{1}{2} + \frac{2}{n^2})} \nrightarrow 1$ as $n \to \infty$.

Hence the pair (f, g) is not compatible satisfying E.A property but not (CLRg).

Example 2.2. Let $(X, M, *)$ be a fuzzy metric space, * being a continuous norm with $X = R$. Define $M(x, y, t) = \frac{t}{t+|x-y|}$ for all x, y in X and $t > 0$. Define mappings f: $X \times X \to X$ and g: $X \to X$ by $f(x, y) = x - y$ and $g(x) = 2x$ for all x, y in X and consider the sequences $x_n = \{\frac{1}{n}\}$ and $y_n = \{-\frac{1}{n}\}$, then f and g satisfy both the. Properties E. A and CLRg

Remark 2.1. From above examples we can say that
1) Weak compatibility does not imply compatibility
2) E.A does not imply (CLRg).
3) E.A and (CLRg) do not imply compatibility.

The next example shows that the maps satisfying (CLRg) property need not be continuous,

Example 2.3. Let $(X, M, *)$ be a fuzzy metric space, * being a continuous norm with $X =$ $[0, \infty)$.Define $M(x, y, t) = \frac{t}{t+|x-y|}$ for all x, y in X and $t > 0$.Define mappings f: $X \times X \to X$ and g: $X \to X$ as follows

$$f(x, y) = \begin{cases} x + y & if\ x \in [0, 1), y \in X \\ \frac{x+y}{2} & if\ x \in [1, \infty), y \in X \end{cases} \quad \text{and } g(x) = \begin{cases} 1 + x & if\ x \in [0, 1) \\ \frac{x}{2} & if\ x \in [1, \infty) \end{cases}$$

We consider the sequences $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{1 + \frac{1}{n}\}$.Then, the maps f and g satisfy (CLRg) property but are not continuous.

III. Main Results

Xin-Qi Hu [4] proved the following result:

Theorem 3.1. Let $(X, M, *)$ be a complete FM-space, where * is a continuous t-norm of H-type. Let f: $X \times X \to X$ and g: $X \to X$ be two mappings and there exists $\phi \in \Phi$ such that (3.1) $M(f(x, y), f(u, v), \phi(t)) \geq M(gx, gu, t) * M(gy, gv, t)$,for all x, y, u, v in X and $t > 0$.

Suppose that $f(X \times X) \subseteq g(X)$ and g is continuous, f and g are compatible. Then, there exists $x \in X$such that $x = g(x) = f(x, x)$, that is, f and g have a common fixed point in X.

Now , we give generalization of Theorem 3.1 as follows.

Theorem 3.2. Let $(X, M, *)$ be a FM - Space, * being continuous t – norm of H-type. Let f: $X \times X \to X$ and g: $X \to X$ be two mappings and there exists $\phi \in \Phi$ satisfying (3.1) and the pair (f, g) satisfy (CLRg) property . Then f and g have a coupled coincidence point in X. Moreover, if (f, g) is weakly compatible there exists a unique point x in X such that $x = f(x, x) = g(x)$.

Proof. Since f and g satisfy CLRg property, there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} g(x_n) = g(p)$, $\lim_{n\to\infty} f(y_n, x_n) = \lim_{n\to\infty} g(y_n) = g(q)$ for some p, q in X.

Step 1. To show that f and g have a coupled coincidence point.

From (3.1), $M(f(x_n, y_n), f(p, q), \phi(t) \geq M(gx_n, g(p), t) * M(gy_n, g(q), t)$

Taking limit as $n \to \infty$, we get , $M(g(p), f(p, q), \phi(t)) = 1$ that is, $f(p, q) = g(p) = x$.

Similarly, $f(q, p) = g(q) = y$. Since f and g are weakly compatible, so that $f(p, q) = g(p) = x$ and $f(q, p) = g(q) = y$ implies $gf(p, q) = f(g(p), g(q))$ and $gf(q, p) = f(g(q), g(p))$ that is $g(x) = f(x, y)$ and $g(y) = f(y, x)$. Hence f and g have a coupled coincidence point.

Step 2. To show that $g(x) = x$, $g(y) = y$.

Since * is a t-norm of H-type, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$\underbrace{(1-\delta) \times (1-\delta) \times \ldots \times (1-\delta)}_{p} \geq (1-\epsilon)$, for all $p \in N$.

Since $\lim_{t\to\infty} M(x, y, t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$M(gx, x, t_0) \geq (1-\delta)$ and $M(gy, y, t_0) \geq (1-\delta)$.

Also, since $\phi \in \Phi$, using condition (ϕ-3), we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0$, there exists $n_0 \in N$ such that $t > \sum_{k=n_0}^{\infty} \phi^k(t_0)$. From (3.2), we have

$M(gx, x, \phi(t_0)) = M(f(x, y), f(p, q), \phi(t_0) \geq M(gx, gp, t_0) * M(gy, gq, t_0) = M(gx,x, t_0) * M(gy, y, t_0)$,

similarly, $M(gy, y, \phi(t_0)) \geq M(gy, y, t_0) * M(gx, x, t_0)$. Therefore, we can get for all $n \in N$,

$M(gx, x, \phi^{n}(t_0)) \geq M(gx, x, \phi^{n-1}(t_0)) * M(gy, y, \phi^{n-1}(t_0))$
$\geq [M(gx, x, t_0)]^{2^{n-1}} * [M(gy, y, t_0)]^{2^{n-1}}$,

thus, we have , $M(gx, x, t) \geq M(gx, x, \sum_{k=n_0}^{\infty} \phi^{k}(t_0)) \geq M(gx, x, \phi^{n_0}(t_0))$

$[M(gx, x, t_0)]^{2^{n_0-1}} \times [M(gy, y, t_0)]^{2^{n_0-1}} \geq$

$\underbrace{(1-\delta) \times (1-\delta) \times \ldots \times (1-\delta)}_{2^{n_0}} \geq (1-\epsilon) \geq$

So, for any $\epsilon > 0$, we have, $M(gx, x, t) \geq (1-\epsilon$, for all $t > 0$ $\Rightarrow g(x) = x$. Similarly, $g(y) = y$.

Step 3. Next we shall show that $x = y$. Using condition (3.1), we have

$M(x, y, \phi(t_0)) = M(f(p, q), f(q, p), \phi(t_0))$
$\geq M(gp, gq, t_0) * M(gq, gp, t_0) = M(x, y, t_0) * M(y, x, t_0)$,

thus, we have $M(x, y, t) \geq M(x, y, \sum_{k=n_0}^{\infty} \phi^{k}(t_0))$
$\geq M(x, y, \phi^{n_0}(t_0))$

$\geq [M(x, y, t_0)]^{2^{n_0-1}} \times [M(y, x, t_0)]^{2^{n_0-1}} \geq$

$\underbrace{(1-\delta) \times (1-\delta) \times \ldots \times (1-\delta)}_{2^{n_0}} \geq (1-\epsilon)$,

which implies that $x = y$. thus, we have proved that f and g have a common fixed point x in X , uniqueness of x follows easily from (3.1) and hence the theorem .

Next, we give an example in support of theorem 3.2.

Example 3.1. Let $X = [0, 2]$, $a \times b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \begin{cases} t^{|x-y|}, t \neq 0 \\ 0, t = 0 \end{cases}$. Then $(X, M, *)$ is a Fuzzy Metric space. Define the mappings

$f : X \times X \rightarrow X$ by $f(x, y) = x+y$ and $g : X \rightarrow X$ by $g(x) = \frac{x}{2}$

set $\phi(t) = t$

Consider the sequences $\{x_n\} = \{ \frac{1}{n} : n \in N\}$ and $\{y_n\} = \{- \frac{1}{n} : n \in N\}$, then the pair (f, g) is weakly compatible and satisfies (CLRg) property . We now check the condition (3.1),

$$M(f(x,y), f(u,v), \phi t) = M(x+y, u+v, t) = t^{|x+y-(u+v)|} = t^{|x-u+y-v|}$$
$$= t^{|x-u|} \cdot t^{|y-v|} \geq t^{\frac{1}{2}|x-u|} \cdot t^{\frac{1}{2}|y-v|} = M(gx, gu, t) * M(gy, gv, t)$$

Hence, all the conditions of Theorem 3.2, are satisfied. Thus f and g have a unique common coupled fixed point in X. Indeed, $x = 0$ is the unique common fixed point of f and g.

Corollary 3.2. Let $(X, M, *)$ be a FM - Space, * being continuous t – norm of H-type. Let $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ be two mappings and there exists $\phi \in \Phi$ satisfying (3.1) and (3.2).$f(X \times X) \subseteq g(X)$, (3.3) the pair (f, g) satisfy E.A. property.

if range of one of the maps f or g is a closed subspace of X, then f and g have a unique common fixed point in X.

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