A Generalization of the Notion of Connes-Amenability

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***Abstract*—In this paper, we show that the approximate Connes-amenability and approximate amenability of weighted group algebra $\ell^1(G,\omega)$ are the same, where G is a discrete group and ω is a weight function on G.**

***Index Terms*—Connes-amenability, approximate amenability, beurling algebras.**

I. Introduction

The notion of approximate amenability was introduced by F. Ghahramani and R. J. Loy in [3]. The class of approximately amenable Banach algebras is clearly larger than that for the classical amenable Banach algebras introduced by B. E. Johnson in [5], (see [3] and [4]).

A weaker notion of amenability that is called Connes-amenability, which has a natural generalization to the case of dual Banach algebras, was introduced by V. Runde in [6,7].

With above notions in hands, it is normal to think of the concept of approximate Connes-amenability for dual Banach algebras, [2]. There are examples to show that this new notion is not coincide with the mentioned concepts earlier, [2, Example 2.1].

In this short note, we investigate the notion of approximate Connes-amenability for weighted group algebras. We include only the proof of our main Theorem.

Our work is based on [1]. For a discrete group G we show that $\ell^1(G,\omega)$ is approximately Connes- amenable if and only if it is approximately amenable.

A Banach A-bimodule E is called *dual* if there is a closed submodule E_* of E^* such that $E=(E_*)^*$. The dual Banach A-bimodule E is called *normal* if the module actions of A on E are ω^*-continuous. A Banach algebra A is dual if it is dual as a Banach A-bimodule. A dual Banach algebra A is *Connes-amenable* if for every normal, dual Banach A-bimodule E, all bounded, ω^*-continuous derivations $D:A\to E$ are inner. Let $A=(A_*)^*$ be a dual Banach algebra and let $\pi: A\hat{\otimes} A\to A$ be the diagonal operator induced by $a\otimes b\to ab$. Let E be a Banach A-bimodule. We write $\sigma wc(E)$ for the set of all elements $x\in E$ such that the maps $a\mapsto a.x$ and $a\mapsto x.a$ from A into E are ω^*-weak continuous. The space $\sigma wc(E)$ is a closed submodule of E. It is shown that $\pi^* A_* \subseteq \sigma wc(A\hat{\otimes} A)^*$, [8, Corollary 4.6]. Taking adjoint, we can extend π to an A-bimodule homomorphism $\pi_{\sigma wc}$ from $\sigma wc\left(\left(A\hat{\otimes} A\right)^*\right)^*$ to A. A derivation $D:A\to E$ is *approximately inner* if there exists a net $(x_\alpha)_\alpha\subseteq E$, not necessary bounded, such that for every $a\in A$, $Da=\lim_\alpha a.x_\alpha - x_\alpha.a$, the limit being in norm. A dual Banach algebra A is *approximately Connes- amenable* if for every normal, dual Banach A-bimodule E, every ω^*-continuous derivation $D:A\to E$, is approximately inner.

II. The Main Result

For a Banach algebra A, we define the bilinear maps $A^{**}\times A^*\to A^*$ and $A^*\times A^{**}\to A^*$ by

$$\langle a,\Phi.f\rangle=\langle f.a,\Phi\rangle,\ \langle a,f.\Phi\rangle=\langle a.f,\Phi\rangle\quad (a\in A,\ f\in A^*,\ \Phi\in A^{**}).$$

Then we have two algebra products on A^{**}, called the *first* and *second* Arens product respectively. When they are equal, we say that A is *Arens regular*. In this case A^{**} is a dual Banach algebra with predual A^*.

Let G be a group. A function $\omega: G\to(0,\infty)$ is called a weight if $\omega(st)\le\omega(s)\omega(t)$, for each $s,t\in G$. Without loss of generality, we may put $\omega(e)=1$, where e is the identity of G.

For a weight ω on a group G, the Banach space $\ell^1(G,\omega)$ is called a Beurling algebra. Following [1], we consider $\ell^1(G,\omega)$ as the Banach space $\ell^1(G)$ with the product $\delta_g *_\omega \delta_h=\delta_{gh}\Omega(g,h)$ where $\Omega(g,h)=\omega(gh)/\omega(g)\omega(h)$, for $g,h\in G$, and extend $*_\omega$ to $\ell^1(G)$ by linearity and continuity.

For a group G, the spaces $c_0(G)$ and $\ell^\infty(G)$ are known. We write $(u_s)_{s\in G}$ for the standard unit vector basis of $c_0(G)$, so that $\langle u_s,\delta_t\rangle=\delta_{s,t}$, the *Kronecker delta*, for $s,t\in G$. It is known that $\ell^1(G,\omega)$ is a dual Banach algebra with predual $c_0(G)$.

Theorem 1.1. Let G be discrete group, let ω be a weight on G and let $A=\ell^1(G,\omega)$. Then the following are equivalent:

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(i) A is approximately amenable.

(ii) There is a net $(M_\alpha)_\alpha \subseteq (A\hat{\otimes} A)^{**} = \ell^\infty(G\times G)^*$ such that

$$\left\langle \left(f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g)\right)_{(g,h)\in G\times G}, M_\alpha \right\rangle \to 0, \quad (1)$$

for each $k\in G$, uniformly for all $f\in ball\ell^\infty(G\times G)$, and

$$\left\langle \left(f_{gh}\Omega(g,h)_{(g,h)\in G\times G}, M_\alpha \right)\right\rangle \to f_e,$$

$$\left(f = (f_g)_{g\in G} \in \ell^\infty(G)\right). \ \left\langle \left(f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g)\right)_{(g,h)\in G\times G}, M_\alpha \right\rangle \to 0,$$

(iii) There is a net $(M'_\alpha)_\alpha \subseteq (A\hat{\otimes} A)^{**} = \ell^\infty(G\times G)^*$ such that

$$\left\langle \left(f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g)\right)_{(g,h)\in G\times G}, M'_\alpha \right\rangle \to 0, \quad (1)$$

for each $k\in G$, uniformly for all $f\in ball\,\ell^\infty(G\times G)$, and

$$\left\langle \left(f_{gh}\Omega(g,h)_{(g,h)\in G\times G}, M'_\alpha \right)\right\rangle = f_e, \quad \left(f = (f_g)_{g\in G} \in \ell^\infty(G)\right). \quad (2)$$

Proposition 1.2. Let G be a discrete group, let ω be a weight on G and let $A=\ell^1(G,\omega)$. Then the following are equivalent:

(i) A is approximately Connes-amenable, with respect to the predual $c_0(G)$.

(ii) There is a net $(M_\alpha)_\alpha \subseteq (A\hat{\otimes} A)^{**} = \ell^\infty(G\times G)^*$ such that

$$\left\langle \left(f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g)\right)_{(g,h)\in G\times G}, M_\alpha \right\rangle \to 0, \quad (1)$$

for each $k\in G$, uniformly for all $f\in ball\,\ell^\infty(G\times G)$, which are such that the maps $T\in L(A,A^*)$ defined by $\langle \delta_h, T(\delta_g)\rangle = f(g,h)$ for $g,h\in G$, satisfy the conclusions of Theorem 1.1, and

$$\left\langle \left(f_{gh}\Omega(g,h)_{(g,h)\in G\times G}, M_\alpha \right)\right\rangle \to \langle f, e\rangle, \quad (2)$$

uniformly for all $f\in ball\, c_0(G)$.

(iii) There is a net $(M'_\alpha)_\alpha \subseteq (A\hat{\otimes} A)^{**} = \ell^\infty(G\times G)^*$ such that

$$\left\langle \left(f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g)\right)_{(g,h)\in G\times G}, M'_\alpha \right\rangle \to 0, \quad (1)$$

for each $k\in G$, uniformly for all $f\in ball\ell^\infty(G\times G)$, which are such that the maps $T\in L(A,A^*)$ defined by $\langle \delta_h, T(\delta_g)\rangle = f(g,h)$ for $g,h\in G$, satisfy the conclusions of Theorem 1.1, and

$$\left\langle \left(f_{gh}\Omega(g,h)_{(g,h)\in G\times G}, M'_\alpha \right)\right\rangle \to \langle f, e\rangle, \quad \left(f = (f_g)_{g\in G} \in c_0(G)\right). \quad (2)$$

Let G be a discrete group and let $h\in G$. Following Daws as in [1], we define $J_h:\ell^\infty(G)\to\ell^\infty(G)$ by

$$J_h(f) = \left(f_{hg}\,\Omega(h,g)\,\omega(h)\,\Omega(g^{-1},h^{-1})\,\omega(h^{-1})\right)_{g\in G}, \quad \left(f = (f_g)_g \in \ell^\infty(G)\right)$$

It is clear that $\| J_h(f)\| \le \| f\|$, so that J_h is bounded.

Theorem 1.3. Let G be a discrete group, let ω be a weight on G and let $A=\ell^1(G,\omega)$. Consider the following:

(i) A is approximately Connes-amenable, with respect to the predual $c_0(G)$.

(ii) A is approximately amenable.

(iii) There is a net $(N_\alpha)_\alpha \subseteq \ell^\infty(G)^*$ such that

$$\left\langle \left(\Omega(g,g^{-1})\right)_g, N_\alpha \right\rangle \to 1, \quad (1)$$

and

$$J_k^*(N_\alpha) - N_\alpha \to 0, \qquad (k\in G). \quad (2)$$

(iv) There is a net $(N'_\alpha)_\alpha \subseteq \ell^\infty(G)^*$ such that

$$\left\langle \left(\Omega(g,g^{-1})\right)_g, N'_\alpha \right\rangle = 1, \quad (1)$$

and

$$J_k^*(N'_\alpha) - N'_\alpha \to 0, \qquad (k\in G). \quad (2)$$

(v) A^{**} is approximately Connes-amenable.

Then, $(i)\Leftrightarrow(ii)\Leftrightarrow(iii)\Leftrightarrow(iv)$.

Moreover, if A is Arens regular, then $(v)\Rightarrow(i)$.

Proof. $(ii)\Rightarrow(i)$ and $(iv)\Rightarrow(iii)$ are clear. If A is Arens regular, we have the implication $(v)\Rightarrow(i)$.

$(i)\Rightarrow(iv)$: Suppose that the net $(M'_\alpha)_\alpha \subseteq \ell^\infty(G\times G)^*$ are given as in part (iii) of Proposition 1.2. Define $\phi:\ell^\infty(G)\to\ell^\infty(G\times G)$ by putting $\langle \delta_{(g,h)}, \phi(f)\rangle = f_g$ for $g=h^{-1}$, and otherwise equals zero.Let $N'_\alpha = \phi^*(M'_\alpha)$. Then we have

$$\phi\left(\left(\Omega(g,g^{-1})\right)_g\right) = \left(\delta_{gh,e}\Omega(g,h)\right)_{(g,h)\in G\times G}$$

where δ is the Kronecker delta. So that

$$\left\langle \left(\Omega\left(g, g^{-1}\right)\right)_g, N'_\alpha \right\rangle = \left\langle \left(\delta_{gh,e}\Omega(g,h)\right)_{(g,h)\in G\times G}, M'_\alpha \right\rangle = \left\langle \left(\delta_{gh,e}\right)_{(g,h)\in G\times G}, \delta_e \right\rangle = \delta_{e,e} = 1,$$

by condition (2) of part (iii) of Proposition 1.2.

Fix $k \in G$ and $f \in \ell^\infty(G)$. Define $F : G\times G \to C$ by

$$F(g,h) = \delta_{gh,k}\ f_g\ \omega(k)\,\omega\left(hk^{-1}\right)\omega(h)^{-1} \qquad (g,h\in G)$$

It is clear that F is bounded and $\| F \|_\infty \leq \| f \|_\infty\ \omega(k)\,\omega\left(k^{-1}\right)$. Let T be the operator associated with F. It is easy to check that F satisfies the conditions of Proposition 1.2. Notice that

$$\left\langle \delta_{(g,h)}, \phi\left(J_k(f)\right)\right\rangle = \delta_{gh,e}\ f_{kg}\ \omega(kg)\,\omega(g)^{-1}\ \omega\left(g^{-1}k^{-1}\right)\omega\left(g^{-1}\right)^{-1}$$

thus we have

$$\| J_k^*\left(N'_\alpha\right) - N'_\alpha \| = \sup\left\{ \left| \left\langle \left(F(hk,g)\,\Omega(h,k) - F(h,kg)\,\Omega(k,g)\right)_{(g,h)}, M'_\alpha \right\rangle \right| : f \in ball\ \ell^\infty(G)\right\}$$

so that $J_k^*\left(N'_\alpha\right) - N'_\alpha \to 0$, by condition (1) of part (iii) of Proposition 1.2.

$(iii) \Rightarrow (ii)$: To show that A is approximately amenable, we show that part (ii) of Theorem 1.1 is satisfied. Define $\psi : \ell^\infty(G\times G) \to \ell^\infty(G)$ by $\left\langle \delta_g, \psi(F)\right\rangle = F\left(g, g^{-1}\right)$, for each $F \in \ell^\infty(G\times G)$ and $g \in G$. Let $(N_\alpha)_\alpha$ be as part (iii). Let $M_\alpha = \psi^*(N_\alpha)$.

For $f = \left(f_g\right)_g \in \ell^\infty(G)$, we have

$$\left\langle \left(f_{gh}\ \Omega(g,h)\right)_{(g,h)}, M_\alpha \right\rangle = \left\langle \left(f_e\ \Omega\left(g,g^{-1}\right)\right)_g, N_\alpha \right\rangle \to f_e$$

so that the condition (2) part (ii) of Theorem 1.1 holds. Let $f : G\times G \to C$ be a bounded function and let $k \in G$. Then

$$\psi\left(\left(f(hk,g)\,\Omega(h,k) - f(h,kg)\,\Omega(k,g)\right)_{(g,h)}\right) = \left(f\left(g^{-1}k, g\right)\Omega\left(g^{-1},k\right) - f\left(g^{-1},kg\right)\Omega(k,g)\right)_g$$

Define $F : G\times G \to C$ by $F(g,h) = f(hk,g)\,\Omega(h,k)$, for each $g,h \in G$. So F is bounded and $\| F \|_\infty \leq \| f \|_\infty$. Then

$$\left\langle \left(f(hk,g)\,\Omega(h,k) - f(h,kg)\,\Omega(k,g)\right)_{(g,h)}, M_\alpha \right\rangle = \left\langle \psi(f), N_\alpha - J_k^*(N_\alpha)\right\rangle$$

Consequently, using condition (2) part (iii), we have established condition (1) part (ii) of Theorem 1.1.

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