Characterization of P-Compactly Packed Modules

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***Abstract*—Let R be a commutative ring with 1, and M is a (left) R–module. We introduce the concepts of (strongly) p-compactly packed submodules as: A proper submodule N of an R-module M is called P-Compactly Packed if for each family $\{N_\alpha\}_{\alpha\in\Lambda}$ of primary submodules of M with $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$, there exists $\alpha_1,\alpha_2,...,\alpha_n \in \Lambda$ such that $N \subseteq \bigcup_{i=1}^{n} N_{\alpha_i}$. If $N \subseteq N_\beta$ for some $\beta \in \Lambda$, then N is called Strongly P-Compactly Packed. In this paper, we list some basic properties of this concept. In addition, the necessary and sufficient conditions for an R−module M to be (strongly) P-Compactly Packed are investigated.We also generalize the Prime Avoidance Theorem for modules that was proved in [7] to the Primary Avoidance Theorem for modules. Furthermore, we find the conditions on an R-module M that make the following important result true, that is for a multiplication Bezout module M, M is strongly P- compactly packed if and only if every primary submodule of M is strongly P- compactly packed.**

***Index Terms*—P-compactly packed submodule, Strongly p-compactly packed submodule, MAXIMAL submodule, bezout module.**

I. Introduction

Zaynab A.A.Al-Ani generalized the concept of compactly packed rings to modules and introduced the definition of compactly packed submodule and strongly compactly packed submodule; a proper submodule N of an R-module M is called compactly Packed if for each family $\{N_\alpha\}_{\alpha\in\Lambda}$ of prime submodules of M with $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$ there exist $\alpha_1,\alpha_2,...,\alpha_n \in \Lambda$ such that $N \subseteq \bigcup_{i=1}^{n} N_{\alpha_i}$. If $N \subseteq N_\beta$ for some $\beta \in \Lambda$ then N is called Strongly Compactly Packed. A module M is said to be Compactly Packed (Strongly Compactly Packed) if every proper submodule of M is compactly packed (or strongly compactly packed) submodule [1].

In this paper, we discuss the situation when the union of a family of primary submodules of M is considered.

C. P. Lu generalized the Prime Avoidance Theorem to modules in terms of prime submodules [5]. We consider a generalization of this theorem to modules in terms of primary submodules.

II. P-Compactly Packed and Strongly P-Compactly Packed Submodules

We introduce the following definition for p-compactly packed submodule and strongly p-compactly packed submodule.

Manuscript received May 10, 2012; revised June 12, 2012

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Definition

A proper submodule N of an R-module M is called P-Compactly Packed if for each family $\{N_\alpha\}_{\alpha\in\Lambda}$ of primary submodules of M with $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$, there exist $\alpha_1,\alpha_2,...,\alpha_n \in \Lambda$ such that $N \subseteq \bigcup_{i=1}^{n} N_{\alpha_i}$.

If $N \subseteq N_\beta$ for some $\beta \in \Lambda$, then N is called Strongly P-Compactly Packed.

A module M is said to be P-Compactly Packed (Strongly P-Compactly Packed) if every proper submodule of M is p-compactly packed (strongly p-compactly packed).

It is clear every strongly p-compactly packed submodule is a p-compactly packed submodule but the converse is not true is general, as is seen by the following example.

Example

Let V be a vector space of dimension greater than 2 over the field $F = Z/2Z$. Then every subspace of V is prime, so every subspace of V is primary. Let e_1 and e_2 be distinct vectors of a basis for V, $V_1 = e_1F$, $V_2 = e_2F$, $V_3 = (e_1 + e_2)F$, and $L = \{0, e_1, e_2, e_1 + e_2\} = V_1 \cup V_2 \cup V_3$ is an efficient union of three primary submodules with $\sqrt{[V_i : M]} = (0)$, but $L \not\subset V_i$ for every $i = 1,2,3$.

In the following we give a condition under which the converse holed. For that we give a generalization of the prime avoidance theorem [5] in terms of primary submodules.

Definition

Let $L_1, L_2,..., L_n$ be submodules of an R-module M. We call a covering $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$ efficient if no L_k is superflous. Analogously we shall say $L = L_1 \cup L_2 \cup ... \cup L_n$ is an efficient union if none of the L_k's may be excluded.

Any cover or union consisting of submodules of M can be reduced to an efficient one called an efficient reduction by deleting any unnecessary submodules. A covering of a submodule by two submodules of a module is never efficient. Thus $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$ may be possibly an efficient covering only when $n = 1$ or $n > 2$ [6].

Proposition

Let $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$ be an efficient covering consisting of submodules of an R-moduleM where $n > 2$. If $\sqrt{[L_j : M]} \not\subset \sqrt{[L_k : M]}$ for every $j \neq k$, then no L_k for

$k=\{1,...,n\}$ is a primary submodule of M.

Proof. Since $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$ is an efficient covering, then $L = (L \cap L_1) \cup (L \cap L_2) \cup ... \cup (L \cap L_n)$ is an efficient union. Hence for every $k \leq n$ there exists an element $e_k \in L - L_k$. Moreover, by Lemma (??), $\bigcap_{j \neq k} (L \cap L_j) = \bigcap_{j=1}^{n} (L \cap L_j) \subseteq (L \cap L_k)$ that is

$\bigcap_{j \neq k} (L \cap L_j) \subseteq (L \cap L_k)$. Now, for every $j \neq k$, by hypothesis, $\sqrt{[L_j : M]} \not\subset \sqrt{[L_k : M]}$ so that there exists $s_j \in \sqrt{[L_j : M]}$ but $s_j \notin \sqrt{[L_k : M]}$. Therefore there exists $t_j \in Z^+$ such that $s_j^{t_j} M \subseteq L_j$. Let $t = \prod_{j \neq k} t_j = t_1..t_{k-1}t_{k+1}..t_n$. Suppose that some L_k is a primary submodule so $\sqrt{[L_k : M]}$ is a prime ideal of R. Let $s = \prod_{j \neq k} s_j = s_1..s_{k-1}s_{k+1}..s_n$. So $s^t M \subseteq L_j$ for every $j \neq k$. But $s \notin \sqrt{[L_k : M]}$. Consequently $s^t e_k \in (L \cap L_j)$ for every $j \neq k$. But $s^t e_k \notin (L \cap L_k)$. But this contradicts the fact that $\bigcap_{j \neq k} (L \cap L_j) \subseteq (L \cap L_k)$. Therefore L_k is not primary submodule.

Theorem (The Primary Avoidance Theorem)

Let M be an R-module , $L_1, L_2, ..., L_n$ a finite number of submodules of M and L a submodule of M such that $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$ assume that at most two of the $L_i's$ are not primary submodules and that $\sqrt{[L_j : M]} \not\subset \sqrt{[L_k : M]}$ whenever $j \neq k$ then $L \subseteq L_k$ for some k.

Proof. For the given covering $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$, let $L \subseteq L_{i1} \cup L_{i_2} \cup ... \cup L_{i_m}$ be an efficient reduction, then $1 \leq m \leq n$ *and* $m \neq 2$. If $m \rangle 2$ there exists at least one L_{ij} which is primary. In view of proposition (1.4) this is impossible as $\sqrt{[L_j : M]} \not\subset \sqrt{[L_k : M]}$ if $j \neq k$. Hence $m = 1$, thus $L \subseteq L_k$ for some k.

The condition $\sqrt{[L_j : M]} \not\subset \sqrt{[L_k : M]}$ if $j \neq k$ in the statement of the theorem is essential as is seen in example (1.2) If N is a p-compactly packed submodule of an R-moduleM, such that whenever $H \neq K$, then $\sqrt{[H : M]} \not\subset \sqrt{[L : M]}$ for every proper submodules H *and* L *of* M, then by the primary avoidance theorem, N is a strongly p-compactly packed submodule.

Recall that J(M) denotes the Jacobson Radical of M [4, p.55]. The following proposition shows that p-compactly packed modules which have J (M) $\neq$ M, satisfies a certain kind of ascending chain condition.

Proposition

Let M be a p-compactly packed R-module with J (M) $\neq$ M, then M satisfies the ascending chain condition for primary submodules.

Proof. $N_1 \subseteq N_2 \subseteq N_3 \subseteq ...$ be an ascending chain of primary submodules of M. Let $N = \bigcup_i N_i$. We claim that $N \neq M$. In fact if $N = M$ and H is a maximal submodule of M then $H \neq \bigcup_i N_i$, so there exists $n_1, n_2, ..., n_k$ such that $H \subseteq \bigcup_{i=1}^{k} N_{n_i}$, and since $N_1 \subseteq N_2 \subseteq N_3 \subseteq ...$ is an ascending chain, so there exists $m \in \{1, ..., k\}$ such that $\bigcup_{i=1}^{k} N_{n_i} = N_{n_m}$ so $H \subseteq N_{n_m}$, then $H = N_{n_m}$, and consequently $M = \bigcup_i N_i = N_{n_m}$ which is a contradiction. So N is a proper submodule of M, thus there exists $n_1, n_2, ..., n_k$ such that $N \subseteq \bigcup_{i=1}^{k} N_{n_i}$, and since $N_1 \subseteq N_2 \subseteq N_3 \subseteq ...$ is an ascending chain, so there exists $m \in \{1, ..., k\}$ such that $\bigcup_{i=1}^{k} N_{n_i} = N_{n_m}$ that is $\bigcup_i N_i \subseteq N_{n_m}$ so $N_1 \subseteq N_2 \subseteq N_3 \subseteq ... N_{n_m}$. Therefore M satisfies the ascending chain condition on primary submodules.

Since finitely generated or multiplication module has a maximal submodule, the following corollary follows directly from the previous proposition.

Corollary

If M is a generated or multiplication p-compactly finitely module, then M satisfies the ascending chain condition for primary submodules.

The following proposition and theorem give characterizations of strongly p-compactly packed modules. Recall that the primary radical of a submodule N of an R-module M, denoted by $prad_M(N)$ is defined as the intersection of all primary submodules of M which contain N. If there exists no primary submodule of M containing N, we put $prad_M(N)$ =M [7].

A proper submodule N of an R-module M with $prad_M(N)$ = N will be called P-Radical Submodule [7].

Proposition

Let M be an R-module. M is strongly p-compactly packed if and only if every p-radical submodule of M is the primary radical of a cyclic submodule of it.

Proof. Let N be a p-radical submodule of M such that N is not the primaryradical of a cyclic submodule of it, thus for each $m \in N, N \neq prad_M(\langle m \rangle)$. So there exists a primary submodule $L_m \supseteq \langle m \rangle$ but $N \not\subset L_m$.

Thus

$N = \bigcup_{m \in N} \langle m \rangle \subseteq \bigcup_{m \in N} L_m$ *for* $\langle m \rangle \subseteq L_m \not\subset N$. *That is* $L_m \not\subset N$ *for each* $m \in N$.

This contradicts that M is strongly p-compactly packed module. Conversely, let $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$ where N_α is a primary submodule of

M for each $\alpha \in \lambda$ and $N = prad_M(\langle m \rangle)$ for some $m \in N$. Since $m \in N$, $m \in \bigcup_{\alpha\in\Lambda} N_\alpha$, so there exists

$\beta \in \Lambda$ such that $m \in N_\beta$, hence $\langle m \rangle \subseteq N_\beta$, so $prad_M(\langle m \rangle) \subseteq N_\beta$, that is $N \subseteq N_\beta$. Therefore M is a strongly p-compactly packed module. ▪

Theorem

Let M be an R-module. The following statements are equivalent:-

1) M is a strongly p-compactly packed module.
2) For every proper submodule N of M, there exists $m \in N$ such that $prad_M(N) = prad_M(\langle m \rangle)$.
3) For every proper submodule N of M, if $\{N_\alpha\}_{\alpha\in\Lambda}$ is a family of submodules of M, such that $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$, then there exists $\beta \in \Lambda$ such that $N \subseteq prad_M(N_\beta)$.
4) For every proper submodule N of M, if $\{N_\alpha\}_{\alpha\in\Lambda}$ is a family of p-radical submodules of M, with $N \subseteq \{N_\alpha\}_{\alpha\in\Lambda}$ there exists $\beta \in \Lambda$ such that $N \subseteq N_\beta$.

Proof. $(1) \Rightarrow (2)$: By the same argument of the proof of (1.8).

$(2) \Rightarrow (3)$: Let N be a proper submodule of M and $\{N_\alpha\}_{\alpha\in\Lambda}$ be a family of submodules of M, such that $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$. By (2) there exists $m \in N$ such that $prad_M(N) = prad_M(\langle m \rangle)$. Since $m \in \bigcup_{\alpha\in\Lambda} N_\alpha$, it follows

that there exists $\beta \in \Lambda$ such that $m \in N_\beta$ hence $\langle m \rangle \subseteq N_\beta$, so $N \subseteq prad_M(N) = prad_M(\langle m \rangle) \subseteq prad_M(N_\beta)$.

$(3) \Rightarrow (4)$: It follows directly from the definition of P-radical submodule N.

$(4) \Rightarrow (1)$: It is trivial.

In what follows we give a proposition which gives information about a strongly p-compactly packed module with $J(M) \neq M$.

Proposition

Let M be a strongly p-compactly packed R-module such that $J(M) \neq M$. Then M satisfies the ascending chain condition for P-radical submodules.

Proof. Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots$ be an ascending chain of primary p-radical submodules of M. Let $L = \bigcup_i N_i$, then L is a submodule of M. We claim that L is a proper submodule of M. In fact if $L = M$ and H a maximal submodule of M, so $H \not\subset \bigcup_i N_i$ then by theorem ((1.9)(iv)) there exists j such that $H \subseteq N_j$ and since H is a maximal submodule $H = N_j$ and this implies $\bigcup_i N_i \subseteq N_j$ that is $M \subseteq N_j$ which is a contradiction. So L is a proper submodule of M and by theorem (1.9) there exists j such that $L \subseteq N_j$ so $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots N_j$ that is M satisfies the ascending chain condition for p-radical submodules.

The following is an immediate consequence of proposition (1.10).

Corollary

Let M be a finitely generated or multiplication strongly p-compactly packed R-module, then M satisfies the ascending chain condition for p-radical submodules.

Recall that an R-module M is called Bezout Module if every finitely generated submodule of M is cyclic.

In the following proposition we give a condition for the converse of proposition (1.10) to hold.

Proposition

Let M be a Bezout R-module. If M satisfies the ascending chain condition for P-radical submodules, then M is strongly p-compactly packed module.

Proof. Let N be a proper submodule of M, it is easy to show that there exists a finitely generated submodule L of N such that $prad_M(N) = prad_M(L)$ But M is Bezout module so L is a cyclic submodule, there exists $m \in L$, such that $L = \langle m \rangle$, this implies $m \in N$ and $prad_M(N) = prad_M(\langle m \rangle)$ therefore by theorem (1.9), M is a strongly p-compactly packed module.

Now, we give a characterization of a strongly p-compactly packed finitely generated or multiplication module.

Proposition

Let M be a multiplication or finitely generated R-module. If we have one of the following:

1) M is a cyclic module .
2) M is a Bezout module .
3) R is a Bezout ring .

Then M is a strongly p-compactly packed module if and only if every primary submodule of M is a strongly p-compactly packed submodule.

Proof. Suppose that every primary submodule of M is strongly p-compactly packed. Let N be a proper submodule of M such that $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$ where N_α is a primary submodule of M for each $\alpha \in \Lambda$. Assume $\bigcup_{\alpha\in\Lambda} N_\alpha = M$. Then L is strongly p-compactly packed and since $N \subseteq L \subsetneq M = \bigcup_{\alpha\in\Lambda} N_\alpha$, so there exists $\beta \in \Lambda$

such that $L \subseteq N_\beta$, hence $N \subseteq L \subseteq N_\beta$. Now if $\bigcup_{\alpha\in\Lambda} N_\alpha \neq M$, let $S^* = M - \bigcup_{\alpha\in\Lambda} N_\alpha$ and $S = R - \bigcup_{\alpha\in\Lambda} \sqrt{[N_\alpha : M]}$

so S^* is an S-closed subset of M and since $N \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$ it follows $N \subseteq M - S^*$,so there exists a submodule L

Maximal in $M - S^*$ and contains N [1, p. 75], L is a prime [1, p. 61], so primary submodule, but $L \subseteq \bigcup_{\alpha\in\Lambda} N_\alpha$ (because $L \subseteq M - S^*$) so there exists $\beta \in \Lambda$ such that $L \subseteq N_\beta$, hence $N \subseteq L \subseteq N_\beta$ Therefore M is a strongly p-compactly packed module. The converse is trivial.

In the remainder of this section we shall investigate the relation between the strongly p-compactly packed modules, p-compactly packed modules and the modules of fractions.

Our next result has some interest in itself.

Lemma

Let M be an R-module and S a multiplicatively closed set in R. If W is a primary submodule of the R_S−module M_S, then $\phi^{-1}(W)$ is a primary submodule of M.

Proof. Suppose that W is a primary submodule of M_S. First to show that is proper submodule of M, it is sufficient to show $[\phi^{-1} : M] \bigcap S = \varphi$. Suppose $r \in [\phi^{-1} : M] \bigcap S$, thus $r \in S$ and $rm \in \phi^{-1}(W)$ for all $m \in M$, $\phi(rm) = \frac{rm}{1} \in W$, for all $m \in M$. Let $\frac{a}{t} \in M_S$, so $\frac{a}{t} = \frac{ra}{rt} = \frac{ra}{1}\cdot\frac{1}{rt} \in W$, thus $M_S \subseteq W$ which is contradiction.

Now to show $\phi^{-1}(W)$ is primary submodule, let $r \in R$, $m \in M$ such that $rm \in \phi^{-1}(W)$ so $\phi(rm) \in W$, $\frac{rm}{1} = \frac{r}{1}\cdot\frac{m}{1} \in W$ but W is primary submodule of M_S, hence either $\frac{m}{1} \in W$ or $\phi(m) = \frac{m}{1} \in W$ this implies $m \in \phi^{-1}(W)$ or $\frac{r}{1} \in \sqrt{[W : M_S]}$ so there exists $n \in Z^+$ such that $\frac{r^n}{1}\cdot\frac{m}{s} \in W$ for all $\frac{m}{s} \in M_S$.

Therefore $\phi(r^n m) = \frac{r^n m}{1} = \frac{r^n ms}{s} = \frac{r^n m}{s}\cdot\frac{s}{1} \in W$, hence $r^n m \in \phi^{-1}(W)$ for all $m \in M$, thus $r \in \sqrt{[\phi^{-1}(W) : M]}$, therefore $\phi^{-1}(W)$ is primary.

Now, we look at the relation between strongly p-compactly packed module M, and the module of fractions M_S.

Proposition

Let M be an R-module and S a multiplicatively closed set in R. If M is strongly p-compactly packed R-module then M_S is strongly p-compactly packed R_S-module.

Proof. Suppose $H \subseteq \bigcup_{\alpha\in\Lambda} W_\alpha$ where H is a proper submodule of M_S and W_α is a primary submodule of M_S for each $\alpha \in \Lambda$. Hence $\phi^{-1}(H) \subseteq \phi^{-1}(\bigcup_{\alpha\in\Lambda} W_\alpha)$. So $\phi^{-1}(H) \subseteq \bigcup_{\alpha\in\Lambda} \phi^{-1}(W_\alpha)$ By Lemma (1.14) $\phi^{-1}(W_\alpha)$ is a primary submodule of M, there exists $\beta \in \Lambda$ such that $\phi^{-1}(H) \subseteq \phi^{-1}(W_\beta)$ hence $(\phi^{-1}(H))_S \subseteq (\phi^{-1}(W_\beta))_S$. We will show that $(\phi^{-1}(K))_S = K$ for every submodule K of M_S. Let $\frac{x}{s} \in (\phi^{-1}(K))_S$ where $x \in \phi^{-1}(K)$ and $s \in S$. So $\phi(x) \in K$, that is $\frac{x}{1} \in K$, hence $\frac{x}{1}\cdot\frac{1}{s} = \frac{x}{s} \in K$, so $(\phi^{-1}(K))_S \subseteq K$. Now let $\frac{x}{s} \in K$, thus $\frac{x}{s}\cdot\frac{s}{1} \in K$, hence $\frac{x}{1} \in K$ that is $\phi(x) \in K$ so $x \in \phi^{-1}(K)$, thus $\frac{x}{s} \in (\phi^{-1}(K))_S$ therefore $K \subseteq (\phi^{-1}(K))_S$, consequently $K = (\phi^{-1}(K))_S$ for all $K \subseteq M_S$. It follows $H \subseteq W_\beta$. Hence M_S is a strongly p-compactly packed module.

Turning now to the relation between p-compactly packed module M and the module of fractions M_S.

Proposition

Let M be an R-module and S a multiplicatively closed set in R. If M is p-compactly packed R-module then M_S is p-compactly packed R_S-module.

Proof. Let $H \subseteq \bigcup_{\alpha\in\Lambda} W_\alpha$ H is a proper submodule of M_S and W_α is a primary submodule of M_S for each $\alpha \in \Lambda$.

Hence $\phi^{-1}(H) \subseteq \phi^{-1}(\bigcup_{\alpha\in\Lambda} W_\alpha)$. So $\phi^{-1}(H) \subseteq \bigcup_{\alpha\in\Lambda} \phi^{-1}(W_\alpha)$ By Lemma (1.14) $\phi^{-1}(W_\alpha)$ is a primary submodule of M,

there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $\phi^{-1}(H) \subseteq \bigcup_{i=1}^n \phi^{-1}(W_{\alpha i})$ hence $(\phi^{-1}(H))_S \subseteq \bigcup_{i=1}^n ((\phi^{-1}(W_\beta)))_S = \bigcup_{i=1}^n (\phi^{-1}(W_{\alpha_i}))_S$.

Now, as in the proof of proposition (1.15), $H \subseteq \bigcup_{\alpha\in\Lambda} W_\alpha$.

Therefore M_S is p-compactly packed module.

The converses of the last two propositions are not true in general as is seen in the following example:

Example

Let X be an infinite set. Let R be the ring $(P(X), \Delta, \bigcap)$ which is a Bolean ring so it is regular.

Let T = {H|H is a finite subset of X}, so T is non-maximal ideal of P(X), and for any $H \in T$ we have $\langle H \rangle$ is a radical ideal, since every proper ideal in a regular ring is radical ideal. This implies that $\langle H \rangle = \bigcap$ {P|P is a prime ideal contains H}. It is easy to show that every primary ideal L of P(X) is prime. This implies that $prad_{P(X)}(H) = \langle H \rangle$, since $T \not\subset \langle H \rangle$ for all $H \in T$, that is $T \not\subset prad_{P(X)}(\langle H \rangle)$ for all $H \in T$ so there exists primary ideal P_H such that $P_H \supseteq \langle H \rangle$

but $T \not\subset P_H$. Since $T = \bigcup_{H \in T} \langle H \rangle \subseteq \bigcup_{H \in T} P_H$ So T is not p-compactly packed submodule. So P(X) is not p-compactly packed module.

On the other hand, for any maximal ideal P of R, R_P is a field because R is a regular ring, so R_P is p-compactly packed R_P-module.

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