Abstract—Let $R$ be a commutative ring with 1, and $M$ is a (left) $R$-module. We introduce the concepts of (strongly) $p$-compactly packed submodules as: A proper submodule $N$ of an $R$-module $M$ is called $P$-Compactly Packed if for each family $\{N_\alpha\}_{\alpha \in \Lambda}$ of primary submodules of $M$ with $N \subseteq \bigcup_{\alpha \in \Lambda} N_\alpha$, there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $N \subseteq \bigcup_{i=1}^n N_\alpha_i$. If $N \subseteq N_\beta$ for some $\beta \in \Lambda$, then $N$ is called $\beta$-Compactly Packed. We introduce the concepts of (strongly) $p$-compactly packed submodules as: A proper submodule $N$ of an $R$-module $M$ is called $\beta$-Compactly Packed if for each family $\{N_\alpha\}_{\alpha \in \Lambda}$ of primary submodules of $M$ with $N \subseteq \bigcup_{\alpha \in \Lambda} N_\alpha$, there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $N \subseteq \bigcup_{i=1}^n N_\alpha_i$. If $N \subseteq N_\beta$ for some $\beta \in \Lambda$, then $N$ is called $\beta$-Compactly Packed.

Characterization of P-Compactly Packed Modules

Lamis J. M. Abu Lebda

I. INTRODUCTION

Zaynab A.A. Al-Ani generalized the concept of compactly packed rings to modules and introduced the definition of compactly packed submodule and strongly compactly packed submodule; a proper submodule $N$ of an $R$-module $M$ is called compactly packed submodule if for each family $\{N_\alpha\}_{\alpha \in \Lambda}$ of prime submodules of $M$ with $N \subseteq \bigcup_{\alpha \in \Lambda} N_\alpha$ there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $N \subseteq \bigcup_{i=1}^n N_\alpha_i$. If $N \subseteq N_\beta$ for some $\beta \in \Lambda$ then $N$ is called strongly compactly packed submodule (strongly $\beta$-compactly packed submodule).

In this paper, we list some basic properties of this concept. In addition, we prove the necessary and sufficient conditions for each $\alpha \in \Lambda$. We make the following important result true, that is for each family $\{N_\alpha\}_{\alpha \in \Lambda}$ of primary submodules of $M$ with $N \subseteq \bigcup_{\alpha \in \Lambda} N_\alpha$, there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $N \subseteq \bigcup_{i=1}^n N_\alpha_i$. If $N \subseteq N_\beta$ for some $\beta \in \Lambda$, then $N$ is called Strongly $\beta$-Compactly Packed.

II. P-COMPACTLY PACKED AND STRONGLY P-COMPACTLY PACKED SUBMODULES

We introduce the following definition for $p$-compactly packed submodule and strongly $p$-compactly packed submodule.

Definition

A proper submodule $N$ of an $R$-module $M$ is called $P$-Compactly Packed if for each family $\{N_\alpha\}_{\alpha \in \Lambda}$ of primary submodules of $M$ with $N \subseteq \bigcup_{\alpha \in \Lambda} N_\alpha$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $N \subseteq \bigcup_{i=1}^n N_\alpha_i$. If $N \subseteq N_\beta$ for some $\beta \in \Lambda$, then $N$ is called Strongly $\beta$-Compactly Packed.

A module $M$ is said to be $P$-Compactly Packed (Strongly $P$-Compactly Packed) if every proper submodule of $M$ is $p$-compactly packed (strongly $p$-compactly packed).

It is clear every strongly $p$-compactly packed submodule is a $p$-compactly packed submodule but the converse is not true as general, as is seen by the following example.

Example

Let $V$ be a vector space of dimension greater than 2 over the field $F = Z/2Z$. Then every subspace of $V$ is prime, so every subspace of $V$ is primary. Let $e_1$ and $e_2$ be distinct vectors of a basis for $V$, $V_i = e_iF$, $V = \bigcup_{i=1}^3 V_i$, and $L = \{0, e_1, e_2, e_1 + e_2\} = V_1 \cup V_2 \cup V_3$ is an efficient union of three primary submodules with $\sqrt[3]{V_1 : M} = (0)$, but $L \not\subseteq V_i$ for every $i = 1, 2, 3$.

In the following we give a condition under which the converse holds. For that we give a generalization of the prime avoidance theorem [5] in terms of primary submodules.

Definition

Let $L_1, L_2, \ldots, L_n$ be submodules of an $R$-module $M$. We call a covering $L \subseteq L_1 \cup L_2 \cup \ldots \cup L_n$ efficient if no $L_i$ is superfluous. Analogously we shall say $L = L_1 \cup L_2 \cup \ldots \cup L_n$ is an efficient union if none of the $L_i$'s may be excluded.

Any cover or union consisting of submodules of $M$ can be reduced to an efficient one called an efficient reduction by deleting any unnecessary submodules. A covering of a submodule by two submodules of a module is never efficient. Thus $L \subseteq L_1 \cup L_2 \cup \ldots \cup L_n$ may be possibly an efficient covering only when $n = 1$ or $n \geq 2$ [6].

Proposition

Let $L \subseteq L_1 \cup L_2 \cup \ldots \cup L_n$ be an efficient covering consisting of submodules of an $R$-module $M$ where $n \geq 2$. If $\sqrt[L_1 : M] \not\subseteq \sqrt[L_k : M]$ for every $j \neq k$, then no $L_k$ for...
\(k = \{1, \ldots, n\}\) is a primary submodule of \(M\).

**Proof.** Since \(L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n\) is an efficient covering, then
\[ L = (L \cap L_1) \cup (L \cap L_2) \cup \cdots \cup (L \cap L_n) \]

is an efficient union. Hence for every \(k \leq n\) there exists an element \(e_k \in L - L_k\). Moreover, by Lemma (??), \(\bigcap_{j \neq k} (L \cap L_j) = \bigcap_{j=1}^n (L \cap L_j) \subseteq (L \cap L_k)\) that is
\[ \bigcap_{j \neq k} (L \cap L_j) \subseteq (L \cap L_k). \]

Now, for every \(j \neq k\), by hypothesis, \(\sqrt{[L_j : M]} \cap \sqrt{[L_k : M]}\) so that there exists \(s_j \in \sqrt{[L_j : M]}\) but \(s_j \notin \sqrt{[L_k : M]}\). Therefore there exists \(t_j \in Z^+\) such that \(s_j^{t_j} M \subseteq L_j\). Let
\[ t = \prod \{t_j \mid j \neq k\} = t_1 \cdot t_2 \cdot \cdots \cdot t_n. \]

Suppose that some \(L_k\) is a primary submodule so \(\sqrt{[L_k : M]}\) is a prime ideal of \(R\). Let
\[ s = \prod \{s_j \mid j \neq k\} = s_1 \cdot s_2 \cdot \cdots \cdot s_n. \]

But \(s \notin \sqrt{[L_k : M]}\). Consequently \(s' e_k \in (L \cap L_j)\) for every \(j \neq k\). But \(s' e_k \notin (L \cap L_k)\). This contradicts the fact that \(\bigcap_{j \neq k} (L \cap L_j) \subseteq (L \cap L_k)\). Therefore \(L_k\) is not a primary submodule.

**Theorem (The Primary Avoidance Theorem)**

Let \(M\) be an \(R\)-module, \(L_1, L_2, \ldots, L_n\) a finite number of submodules of \(M\) and \(L\) a submodule of \(M\) such that \(L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n\) assume that at most two of the \(L_i\)'s are not primary submodules and that
\[ \sqrt{[L_j : M]} \cap \sqrt{[L_k : M]}\]
whenever \(j \neq k\) then \(L \subseteq L_k\) for some \(k\).

**Proof.** For the given covering \(L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n\), let \(L \subseteq L_i_1 \cup L_i_2 \cup \cdots \cup L_i_n\) be an efficient reduction, then \(1 \leq m \leq n\) and \(m \neq 2\). If \(m > 2\) there exists at least one \(L_{i_j}\) which is primary. In view of proposition (1.4) this is impossible as \(\sqrt{[L_j : M]} \cap \sqrt{[L_k : M]}\) if \(j \neq k\). Hence \(m = 1\), thus \(L \subseteq L_1\) for some \(k\).

The condition \(\sqrt{[L_j : M]} \cap \sqrt{[L_k : M]}\) if \(j \neq k\) in the statement of the theorem is essential as is seen in example (1.2) If \(N\) is a p-compactly packed submodule of a R-module \(M\), such that whenever \(H \neq K\), then \(\sqrt{[H : M]} \cap \sqrt{[L : M]}\) for every proper submodules \(H\) and \(L\) of \(M\), then by the primary avoidance theorem, \(N\) is a strongly p-compactly packed submodule.

Recall that \(J(M)\) denotes the Jacobson Radical of \(M\) [4, p.55]. The following proposition shows that p-compactly packed modules which have \(J(M) \neq M\), satisfies a certain kind of ascending chain condition.

**Proposition**

Let \(M\) be a p-compactly packed \(R\)-module with \(J(M) \neq M\), then \(M\) satisfies the ascending chain condition for primary submodules.

**Proof.** \(N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots\) be an ascending chain of primary submodules of \(M\). Let \(N = \bigcup_j N_j\). We claim that \(N \neq M\). In fact if \(N = M\) and \(H\) is a maximal submodule of \(M\) then \(H \neq \bigcup_j N_j\), so there exists \(n_1, n_2, \ldots, n_k\) such that \(H \subseteq \bigcup_{j=1}^k N_{n_j}\), and since \(N_1 \subseteq N_2 \subseteq \cdots\) is an ascending chain, so there exists \(m \in \{1, \ldots, k\}\) such that \(\bigcup_{j=1}^m N_{n_j} = N_{n_m}\) so \(H \subseteq N_{n_m}\), then \(H = N_{n_m}\), and consequently \(M = \bigcup_{j=1}^m N_j = N_{n_m}\) which is a contradiction. So \(N\) is a proper submodule of \(M\), thus there exists \(n_1, n_2, \ldots, n_k\) such that \(N \subseteq \bigcup_{j=1}^k N_{n_j}\), and since \(N_1 \subseteq N_2 \subseteq \cdots\) is an ascending chain, so there exists \(m \in \{1, \ldots, k\}\) such that \(\bigcup_{j=1}^m N_{n_j} = N_{n_m}\) that is \(\bigcup_{j=1}^m N_{n_j} \subseteq N_{n_m}\) so \(N_1 \subseteq N_2 \subseteq \cdots N_{n_m}\). Therefore \(M\) satisfies the ascending chain condition on primary submodules.

Since finitely generated or multiplication module has a maximal submodule, the following corollary follows directly from the previous proposition.

**Corollary**

If \(M\) is a generated or multiplication p-compactly finitely module, then \(M\) satisfies the ascending chain condition for primary submodules.

The following proposition and theorem give characterizations of strongly p-compactly packed modules. Recall that the primary radical of a submodule \(N\) of an \(R\)-module \(M\), denoted by \(prad(N)\) is defined as the intersection of all primary submodules of \(M\) which contain \(N\). If there exists no primary submodule of \(M\) containing \(N\), we put \(prad(M) = M\) [7].

A proper submodule \(N\) of an \(R\)-module \(M\) with \(prad(M) = N\) will be called P-Radical Submodule [7].

**Proposition**

Let \(M\) be an \(R\)-module. \(M\) is strongly p-compactly packed if and only if every p-radical submodule of \(M\) is the primary radical of a cyclic submodule of it.

**Proof.** Let \(N\) be a p-radical submodule of \(M\) such that \(N\) is not the primaryradical of a cyclic submodule of it, thus for each \(m \in N\), \(\bar{N} \neq N\).

Thus
\[ N = \bigcup_{m \in N} \langle m \rangle \subseteq \bigcup_{m \in N} L_m, \]

for \(m \in N\). That is \(L_m \subseteq N\) for each \(m \in N\).

This contradicts that \(M\) is strongly p-compactly packed.
module. Conversely, let \( N = \bigcup_{\alpha \in \Lambda} N_{\alpha} \) where \( N_{\alpha} \) is a primary submodule of 
\( M \) for each \( \alpha \in \Lambda \) and \( N = \text{prad}_M((m)) \) for some \( m \in N \). Since \( m \in N \), \( m \in \bigcup_{\alpha \in \Lambda} N_{\alpha} \), so there exists 
\( \beta \in \Lambda \) such that \( m \in N_{\beta} \), hence 
\( (m) \subseteq N_{\beta} \), so \( \text{prad}_M((m)) \subseteq N_{\beta} \), that is \( N \subseteq N_{\beta} \). 
Therefore \( M \) is a strongly \( p \)-compactly packed module.

**Theorem**

Let \( M \) be an \( R \)-module. The following statements are equivalent:

1) \( M \) is a strongly \( p \)-compactly packed module.

2) For every proper submodule \( N \) of \( M \), there exists \( m \in N \) such that 
\( \text{prad}_M(N) = \text{prad}_M((m)) \).

3) For every proper submodule \( N \) of \( M \), if \( \{N_{\alpha}\}_{\alpha \in \Lambda} \) is a family of submodules of \( M \), such that 
\( N \subseteq \bigcup_{\alpha \in \Lambda} N_{\alpha} \), then there exists \( \beta \in \Lambda \) such that 
\( N \subseteq \text{prad}_M(N_{\beta}) \).

4) For every proper submodule \( N \) of \( M \), if \( \{N_{\alpha}\}_{\alpha \in \Lambda} \) is a family of \( p \)-radical submodules of \( M \), with \( N \subseteq \bigcup_{\alpha \in \Lambda} N_{\alpha} \) there exists \( \beta \in \Lambda \) such that 
\( N \subseteq N_{\beta} \).

Proof. (1) \( \Rightarrow \) (2): By the same argument of the proof of (1.8).

(2) \( \Rightarrow \) (3): Let \( N \) be a proper submodule of \( M \) and \( \{N_{\alpha}\}_{\alpha \in \Lambda} \) be a family of submodules of \( M \), such that 
\( N \subseteq \bigcup_{\alpha \in \Lambda} N_{\alpha} \). By (2) there exists \( m \in N \) such that 
\( \text{prad}_M(N) = \text{prad}_M((m)) \). Since \( m \in \bigcup_{\alpha \in \Lambda} N_{\alpha} \), it follows 
that there exists \( \beta \in \Lambda \) such that \( m \in N_{\beta} \) hence 
\( (m) \subseteq N_{\beta} \), so \( N \subseteq \text{prad}_M(N) = \text{prad}_M((m)) \subseteq \text{prad}_M(N_{\beta}) \).

(3) \( \Rightarrow \) (4): It follows directly from the definition of \( p \)-radical submodule \( N \).

(4) \( \Rightarrow \) (1): It is trivial.

In what follows we give a proposition which gives information about a strongly \( p \)-compactly packed module with \( J(M) \neq M \).

**Proposition**

Let \( M \) be a strongly \( p \)-compactly packed \( R \)-module such that 
\( J(M) \neq M \). Then \( M \) satisfies the ascending chain condition for \( p \)-radical submodules.

Proof. Let \( N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \) be an ascending chain of primary \( p \)-radical submodules of \( M \). Let \( L = \bigcup_i N_i \), then \( L \) is a submodule of \( M \). We claim that \( L \) is a proper submodule of \( M \). In fact if \( L = M \) and \( H \) a maximal submodule of \( M \), so \( H \not\subseteq \bigcup_i N_i \), then by theorem ((1.9)(iv)) there exists \( j \) such that \( H \not\subseteq N_j \) and since \( H \) is a maximal submodule \( H = N_j \) and this implies 
\( \bigcup_i N_i \subseteq N_j \) that is \( M \subseteq N_j \) which is a contradiction. 
So \( L \) is a proper submodule of \( M \) and by theorem (1.9) there exists \( j \) such that 
\( L \subseteq N_j \) so 
\( N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots N_j \) that is \( M \) satisfies the ascending chain condition for \( p \)-radical submodules.

The following is an immediate consequence of proposition (1.10).

**Corollary**

Let \( M \) be a finitely generated or multiplication strongly \( p \)-compactly packed \( R \)-module, then \( M \) satisfies the ascending chain condition for \( p \)-radical submodules.

Recall that an \( R \)-module \( M \) is called \( p \)-radical Module if 
every finitely generated submodule of \( M \) is cyclic.

In the following proposition we give a condition for the converse of proposition (1.10) to hold.

**Proposition**

Let \( M \) be a \( p \)-radical \( R \)-module. If \( M \) satisfies the ascending chain condition for \( p \)-radical submodules, then \( M \) is strongly \( p \)-compactly packed module.

Proof. Let \( N \) be a proper submodule of \( M \), it is easy to show that there exists a finitely generated submodule \( L \) of \( N \) such that 
\( \text{prad}_M(N) = \text{prad}_M(L) \) But \( M \) is \( p \)-radical module so \( L \) is a cyclic submodule, there exists \( m \in L \), such that 
\( L = \langle m \rangle \), this implies \( m \in N \) and 
\( \text{prad}_M(N) = \text{prad}_M((m)) \) therefore by theorem (1.9), \( M \) is a strongly \( p \)-compactly packed module.

Now, we give a characterization of a strongly \( p \)-compactly packed finitely generated or multiplication module.

**Proposition**

Let \( M \) be a multiplication or finitely generated \( R \)-module. If we have one of the following:

1) \( M \) is a cyclic module.
2) \( M \) is a \( p \)-radical module.
3) \( R \) is a \( p \)-radical ring.

Then \( M \) is a strongly \( p \)-compactly packed module if and only if every primary submodule of \( M \) is a strongly \( p \)-compactly packed submodule.

Proof. Suppose that every primary submodule of \( M \) is strongly \( p \)-compactly packed. Let \( N \) be a proper submodule of \( M \) such that \( N \subseteq \bigcup_{\alpha \in \Lambda} N_{\alpha} \) where \( N_{\alpha} \) is a primary submodule of \( M \) for each \( \alpha \in \Lambda \). Assume \( \bigcup_{\alpha \in \Lambda} N_{\alpha} = M \).

Then \( L \) is strongly \( p \)-compactly packed and since \( N \subseteq L \subseteq M = \bigcup_{\alpha \in \Lambda} N_{\alpha} \), so there exists \( \beta \in \Lambda \).
such that \( L \subseteq N_{\beta} \), hence \( N \subseteq L \subseteq N_{\beta} \). Now if \( \bigcup N_{\alpha} \neq M \), let \( S^* = M - \bigcup N_{\alpha} \) and \( S \supseteq R - \bigcup_{\alpha \in \Lambda} \sqrt{N_{\alpha} : M} \)

so \( S^* \) is an S-closed subset of \( M \) and since \( N \subseteq \bigcup_{\alpha \in \Lambda} N_{\alpha} \) it follows \( N \subseteq M - S^* \), so there exists a submodule \( L \).

Maximal in \( M - S^* \) and contains \( N \) [1, p. 75], \( L \) is a prime [1, p. 61], so primary submodule, but \( S^* = M - \bigcup_{\alpha \in \Lambda} N_{\alpha} \).

Therefore, therefore \( S^* = M - \bigcup_{\alpha \in \Lambda} N_{\alpha} \).

In the remainder of this section we shall investigate the relation between the strongly p-compactly packed modules, \( p \)-compactly packed modules and the modules of fractions.

Our next result has some interest in itself.

**Lemma**

Let \( M \) be an \( R \)-module and \( S \) a multiplicatively closed set in \( R \). If \( W \) is a primary submodule of the \( R_S\)-module \( M_S \), then \( \phi^{-1}(W) \) is a primary submodule of \( M \).

**Proof.** Suppose that \( W \) is a primary submodule of \( M_S \). First to show that is proper submodule of \( M \), it is sufficient to show \( \{ \phi^{-1}(M) \} \cap S = \phi^{-1}(W) \).

Suppose \( r \in \{ \phi^{-1}(M) \} \cap S \) such that \( r \in S \) and \( rm \in \phi^{-1}(W) \) for all \( m \in M \), \( \phi(rm) = \frac{rm}{1} \in W \), for all \( m \in M \). Let \( \frac{a}{t} \in M_S \), so \( \frac{ar}{rt} = \frac{r}{1} \in W \), thus \( M_S \subseteq W \) which is contradiction.

Now to show \( \phi^{-1}(W) \) is primary submodule, let \( r \in R \), \( m \in M \) such that \( rm \in \phi^{-1}(W) \) so \( \phi(rm) = \frac{rm}{1} \in W \), hence either \( \frac{m}{1} \in W \) or \( \phi(m) = \frac{m}{1} \in W \) this implies \( m \in \phi^{-1}(W) \) or \( r \in \sqrt{W : M} \) so there exists \( n \in Z^* \) such that \( \frac{r^nm}{1} \in W \) for all \( \frac{m}{s} \in M_S \).

Therefore \( \frac{r^nm}{1} = \frac{r^ns}{1} \in \sqrt{W : M} \), hence \( \phi^{-1}(W) \) is primary.

Now, we look at the relation between strongly \( p \)-compactly packed module \( M \), and the module of fractions \( M_S \).

**Proposition**

Let \( M \) be an \( R \)-module and \( S \) a multiplicatively closed set in \( R \). If \( M \) is strongly \( p \)-compactly packed \( R \)-module then \( M_S \) is strongly \( p \)-compactly packed \( R_S \)-module.

**Proof.** Suppose \( H \subseteq \bigcup_{\alpha \in \Lambda} W_\alpha \), where \( H \) is a proper submodule of \( M_S \) and \( W_\alpha \) is a primary submodule of \( M_S \) for each \( \alpha \in \Lambda \). Hence \( \phi^{-1}(H) \subseteq \phi^{-1}(\bigcup_{\alpha \in \Lambda} W_\alpha) \).

So \( \phi^{-1}(H) \subseteq \bigcup_{\alpha \in \Lambda} \phi^{-1}(W_\alpha) \).

By Lemma (1.14) \( \phi^{-1}(W_\alpha) \) is a primary submodule of \( M \), there exists \( \beta \in \Lambda \) such that \( \phi^{-1}(H) \subseteq \phi^{-1}(W_\beta) \) hence \( \phi^{-1}(H) \subseteq (\phi^{-1}(W_\beta))_S \).

We will show that \( (\phi^{-1}(K))_S = K \) for every submodule \( K \) of \( M_S \). Let \( \frac{x}{s} \in (\phi^{-1}(K)_S) \), where \( x \in \phi^{-1}(K) \) and \( s \in S \). So \( \frac{x}{s} \in \phi^{-1}(K) \) and \( s \in S \). Therefore \( \frac{x}{s} \in (\phi^{-1}(K)_S) \), hence \( \frac{x}{s} \in (\phi^{-1}(K)_S) \), consequently \( K = (\phi^{-1}(K)_S) \) for all \( K \subseteq M_S \). It follows \( H \subseteq W_\beta \).

Hence \( M_S \) is a strongly \( p \)-compactly packed module.

Turning now to the relation between \( p \)-compactly packed module \( M \) and the module of fractions \( M_S \).

**Proposition**

Let \( M \) be an \( R \)-module and \( S \) a multiplicatively closed set in \( R \). If \( M \) is \( p \)-compactly packed \( R \)-module then \( M_S \) is \( p \)-compactly packed \( R_S \)-module.

**Proof.** Let \( H \subseteq \bigcup_{\alpha \in \Lambda} W_\alpha \), \( H \) is a proper submodule of \( M_S \) and \( W_\alpha \) is a primary submodule of \( M_S \) for each \( \alpha \in \Lambda \).

Hence \( \phi^{-1}(H) \subseteq \bigcup_{\alpha \in \Lambda} \phi^{-1}(W_\alpha) \).

So \( \phi^{-1}(H) \subseteq \bigcup_{\alpha \in \Lambda} \phi^{-1}(W_\alpha) \).

By Lemma (1.14) \( \phi^{-1}(W_\alpha) \) is a primary submodule of \( M \), there exists \( \alpha_1, \alpha_2, ..., \alpha_n \in \Lambda \) such that \( \phi^{-1}(H) \subseteq \bigcup_{\alpha \in \Lambda} \phi^{-1}(W_\alpha) \).

Hence \( \phi^{-1}(H) \subseteq \bigcup_{\alpha \in \Lambda} (\phi^{-1}(W_\alpha)) \).

Now, as in the proof of proposition (1.15), \( H \subseteq \bigcup_{\alpha \in \Lambda} W_\alpha \).

Therefore \( M_S \) is \( p \)-compactly packed module.

The converses of the last two propositions are not true in general as is seen in the following example:

**Example**

Let \( X \) be an infinite set. Let \( R \) be the ring \( (P(X), \Delta, \cap) \) which is a Boolean ring so it is regular.

Let \( T = \{ H | H \subseteq X \} \) so \( T \) is non-maximal ideal of \( P(X) \), and for any \( H \subseteq T \) we have \( \langle H \rangle \) is a radical ideal, since every proper ideal in a regular ring is radical ideal. This implies that \( \langle H \rangle \) is \( \cap \) of \( PP \) which is a prime ideal containing \( H \). It is easy to show that every primary ideal \( L \) of \( P(X) \) is prime. This implies that \( \text{prad}_{P(X)}(L) = \langle H \rangle \), since \( T \subseteq \langle H \rangle \) for all \( H \subseteq T \), that is \( T \subseteq \text{prad}_{P(X)}(\langle H \rangle) \) for all \( H \subseteq T \) so there exists primary ideal \( P_H \) such that \( P_H \subseteq \langle H \rangle \)
but \( T \not\subseteq P_H \). Since \( T = \bigcup_{H \in T} \langle H \rangle \subseteq \bigcup_{H \in T} P_H \) So \( T \) is not \( p \)-compactly packed submodule. So \( P(X) \) is not \( p \)-compactly packed module.

On the other hand, for any maximal ideal \( P \) of \( R \), \( R_P \) is a field because \( R \) is a regular ring, so \( R_P \) is \( p \)-compactly packed \( R_P \)-module.

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