Properties of Lattice Ordered Groups and Γ - Semi Groups

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Abstract—**In this paper we present the properties of lattice ordered groups derived from the properties of partially ordered groups. The notion of Γ-semi groups was introduced by Sen in 1981. The concept of Γ- semigroups is a generalization of the concept of semigroups Many classical notions of semigroups have been extended to Γ-semigroups, (S, Γ , ≤) is called an ordered Γ-semigroup if (S, Γ) is a Γ-semigroup and (S, ≤) is a partially ordered set such that $a \leq b \Rightarrow a\,\gamma\,c \leq b\,\gamma\,c$ and $c\,\gamma\,a \leq c\,\gamma\,b$ for all $a, b, c\,\epsilon\,S$ and $\gamma\,\epsilon\,\Gamma$**

Index Terms—**Partially ordered groups, Lattice ordered groups, semi groups, Γ semi groups and ordered Γ semi groups**

I. INTRODUCTION

A semi group S is said to be partially ordered or a partially ordered semigroup if it is associated with a partial ordering ≤ which is defined by ‘ a ≤ b implies that xay ≤ xby for all x , y in S. The natural partial order which is an obvious partial ordering defined by a ≤ b if and only if a = cb for some $c = c^2$ Є S. This natural partial ordering is compatible with multiplication. Some of the basic properties and results were given by Donald B. McAlister and some of the foundational results are due to A.H. Clifford.

Suppose G is a partially ordered group i.e., G is a group partially ordered by ≤. Now a ≤ b if and only if $1 \leq a^{-1} b$ or equivalently a ≤ b if and only if $1 \leq b\, a^{-1}$.

Let S and **Γ** be non empty sets. If there exists a mapping $S \times \Gamma \times S \rightarrow S$, written (a, γ, b) by $a\,\gamma\,b$, S is called a **Γ**– semigroup if S satisfies $(a\,\gamma\,b)\,\mu\,c = a\,\gamma\,(b\,\mu\,c)$ for all $a, b, c\,\epsilon\,S$ and $\gamma, \mu\,\epsilon\,\Gamma$. Let S be an arbitrary semigroup and **Γ** any nonempty set. Define a mapping $S \times \Gamma \times S \rightarrow S$ by $a\,\gamma\,b = ab$ for all $a, b\,\epsilon\,S$ and $\gamma\,\epsilon\,\Gamma$. It is easy to see that S is a **Γ**-semigroup. Hence a semigroup can be considered to be a **Γ**-semigroup. (S, ≤) is a partially ordered set such that $a \leq b \Rightarrow a\,\gamma\,c \leq b\,\gamma\,c$ and $c\,\gamma\,a \leq c\,\gamma\,b$ for all $a, b, c\,\epsilon\,S$ and $\gamma\,\epsilon\,\Gamma$

II. PROPERTIES OF PARTIAL ORDER GROUPS

Here we consider the set G^+ consisting of elements exceeding the identity 1 and has the following properties:

1) G^+ is a submonoid of G
2) $a\,G^+ = G^+ a$ for each $a \in G$
3) 1 is the only invertible element of G^+

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Proposition 2.1 Let G be a partially ordered group and suppose that $a, b \in G$. Then a and b have a least upper bound $a \vee b$ in G if and only if they have a greatest lower bound$a \wedge b$. This is only possible only when a^{-1} and b^{-1} have a least upper bound. In particular,

$$a \wedge b = a\,(a \vee b)^{-1}\,b$$
$$a \vee b = a\,(a \wedge b)^{-1} b$$
$$a \wedge b = (a^{-1} \vee b^{-1})^{-1}$$
$$a \vee b = (a^{-1} \wedge b^{-1})^{-1} .$$

Also for any $g \in G$,

$$g(a \vee b) = ga \vee gb$$
$$(a \vee b)g = ag \vee bg$$

$$g(a \wedge b) = ga \wedge gb$$
$$(a \wedge b)g = ag \wedge bg$$

Proof. Refer McAlister Lecture notes, University of Lisbon

Corollary 2.2 The following are equivalent for a partially ordered group G.

1) G is a V-semilattice under ≤
2) G is a ∧ -semilattice under ≤
3) a V 1 exists for each a Є G
4) a ∧ 1 exists for each a Є G
5) a V b exists for each a,b Є G^+
6) a ∧ b exists for each a,b Є G^+
7) for each a,b Є G^+ there exists c Є G^+ such that $G^+a \cap G^+b = G^+c$.

If G satisfies one of the conditions in the above corollary, we say that G is a lattice ordered group or simply latticed group.

Definition 2.2.1 A lattice G is called a distributive lattice, if for any a,b,c Є G,

(i) a V (b ∧ c) = (a V b) ∧ (a V c) and (ii) a ∧ (b V c) = (a ∧ b) V (a ∧ c).

Clearly conditions (i) and (ii) are equivalent.

Definition 2.2.2 A lattice G is called a modular lattice, if for any a,b,c Є G such that a ≤ c implies a V (b ∧ c) = (a V b) ∧ c.

Theorem 2.2.3

Let G be a lattice ordered group under a partial order ≤ , then G is a modular lattice under ≤ .

Proof. The proof is obvious, because if G is a lattice ordered group w.r.t a partial order ≤, then G is a distributive lattice under ≤ and every distributive lattice is a modular lattice.

Definition 2.2.4

Two elements a, b of a lattice ordered group G are said to be orthogonal if a ∧ b = 1.

Proposition 2.2.5

Let G be a lattice ordered group and let a,b,c ∈ G. If a ∧ b = 1, then ac ∧ bc = c.

Proof. Since a ∧ b = 1,

$$
\begin{aligned}
ac \wedge bc &= 1\ (ac \wedge bc) \\
&= (a \wedge b)(ac \wedge bc) \\
&= a(ac \wedge bc) \wedge b(ac \wedge bc) \\
&= a^2c \wedge abc \wedge bac \wedge b^2c \\
&= a^2c \wedge abc \wedge b^2c, \text{ since } a \wedge a = a \\
&= c(a(a \wedge b) \wedge b^2) \\
&= c(a \wedge b^2) \\
&= c, \text{ since } a^m \wedge b^n = 1\ \forall m,n \geq 0
\end{aligned}
$$

III. Completely Regular Γ-Semigroups

An element of a Γ- Semigroup S is completely regular if there exists $x \in S$ such that $a = a\alpha x\beta a, a\alpha x = x\beta a$

A Γ-semigroup S is completely regular if all its elements are completely regular

Proposition: If a is an element of a Γ-semigroup S and if $a\epsilon(S\,\Gamma\,a^2)\cap(a^2\,\Gamma\,S)$ then a is contained in the greatest subgroups of S having e as its identity

Proof: If $a\epsilon(S\,\Gamma\,a^2)\cap(a^2\,\Gamma\,S)$, then

$a = x\,\alpha a^2 = a^2\alpha y$ for $x, y\,\epsilon\,S, \alpha\,\epsilon\,\pi$

$\Rightarrow x\alpha a = x\,\alpha(a^2\alpha y) = (x\alpha a^2)\alpha y$

$= (a^2\alpha y)\alpha y(\because x\alpha a^2 = a^2\alpha y\alpha)$

$= a\alpha y(\because a^2\alpha y = a)$

$\therefore x\alpha a = a\alpha y$

Let $e = x\alpha a = a\alpha y$, e being the identity

$a\alpha e = a^2\alpha y = a = x\alpha a^2 = e\alpha a$

$e^2 = (x\alpha a)(a\alpha y) = (x\alpha a^2)\alpha y = a\alpha y = e$

∴ e is idempotent

$\Rightarrow e\epsilon(S\,\Gamma\,a)\cap(a\,\Gamma\,S)$

We know $Ge = \{a\epsilon S/a\epsilon(e\Gamma S)\cap(S\Gamma e), e\epsilon(a\Gamma S)\cap(S\Gamma a)\}$ is the greatest subgroup having e as its identity

Clearly $a\epsilon Ge$

Notation: If a is completely regular element of a Γ -semigroups, we denote by a^{-1}, the inverse of a in the maximal subgroup of S containing 'a'

Definition: A Γ-semigroups S is left cancellative if $x\alpha a = x\alpha b$ implies $a = b$; for any $a, b, x\epsilon S, \alpha\epsilon\Gamma$ cancellative if it is both left and right cancellative

Definition: A Γ-semigroups S is separative if for any x,y ϵS

1) $x^2 = x\alpha y\ and\ y^2 = y\alpha x$ imply $x = y$
2) $x^2 = y\alpha x\ and\ y^2 = x\alpha y$ imply $x = y$

Lemma: In a separative Γ -semigroup S, for any $x, y, a, b\ \epsilon S\ \alpha, \beta\epsilon\Gamma$, the following statements hold

3) $x\alpha a = x\alpha b\ if\ and\ only\ if\ a\alpha x = b\alpha x$
4) $x^2\alpha a = x^2\alpha b\ implies\ x\alpha a = x\alpha b$
5) $x\alpha y\beta a = x\alpha y\beta b\ implies\ y\alpha x\beta a = y\alpha x\beta b$
6)

Proof:

1) If $x\alpha a = x\alpha b$, then

$a\alpha(x\alpha a)\alpha x = a\alpha(x\alpha b)\alpha x, \alpha\epsilon\Gamma$

And $b\alpha(x\alpha a)\alpha x = b\alpha(x\alpha b)\alpha x$

So that $(a\alpha x^2) = (a\alpha x)(b\alpha x)$ and

$(b\alpha x^2) = (b\alpha x)(a\alpha x)$

By separativity, $a\alpha x = b\alpha x$

The opposite implication can be desired easily by using symmetry property

2) If $x^2\alpha a = x^2\alpha b$, then by part (i)

$x\ \alpha\ a\ \alpha\ x = x\ \alpha\ b\ \alpha\ x$

Hence $(a\alpha x)^2 = (a\alpha x)(b\alpha x)$ and

$(b\alpha x)^2 = (b\alpha x)(a\alpha x)$ and thus by separativity $a\alpha x = b\alpha x$

Then by part i), $x\alpha a = x\alpha b$

3) Let $x\alpha y\beta a = x\alpha y\beta b$,

Then $x\alpha y\beta a\delta y = x\alpha y\beta\delta y$

By part i), $y\alpha a\beta y\delta x = y\alpha b\beta y\delta x$

Multiplying by suitable elements on the right as using part i), we obtain the following equalities:

$(a\alpha y\beta x)^2 = (b\alpha y\beta x)(a\alpha y\beta x)$

$(a\alpha y\beta x)(b\alpha y\beta x) = (b\alpha y\beta x)^2$

Which by separating implies $a\alpha y\beta x = b\alpha y\beta x$

But by part i), we have $y\alpha x\beta a = y\alpha x\beta b$

IV. Conclusion

In this paper, we related semigroups with Γ-semigroups and we derived different properties Γ-semigroups by using the properties of semigroups.

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