Two New Integrable Couplings of Soliton Hierarchies with Self-consistent Sources

Sixing Tao and Hui Shi

***Abstract*—A new kind of integrable couplings of soliton hierarchies with self-consistent sources associated with $\widetilde{sl}(4)$ was presented. Based upon this method, two new integrable couplings associated with Li soliton hierarchy and BKK soliton hierarchy with self-consistent sources were obtained by making use of loop algebra $\widetilde{sl}(4)$ respectively. Meanwhile, their Hamiltonian structures of the integrable couplings of Li and BKK soliton hierarchies were obtained by making use of variational identity . The method in this study can be applied to other soliton hierarchies with self-consistent sources.**

***Index Terms*—Li soliton hierarchy, self-consistent sources, integrable couplings, BKK soliton hierarchy**

I. Introduction

Soliton equations with self-consistent sources [1]-[4] has been receiving growing attention in recent years. Physically, the sources may result in solitary waves with a non-constant velocity and therefore lead to a variety of dynamics of physical models. For applications, these kinds of systems are usually used to describe interactions between different solitary waves and are relevant to some problems of hydrodynamics, solid state physics, plasma physics, etc.

Recently, the integrable couplings with self-consistent sources have been receiving growing attention. Yu [5]-[7] has derived some integrable couplings of soliton hierarchy with self-consistent sources such as the Dirac soliton hierarchy with self-consistent sources, the Yang soliton hierarchy with self-consistent sources and the C-KdV soliton hierarchy with self-consistent sources by making use of the loop algebra $\widetilde{sl}(4)$, but there are some errors in [5]-[7] . In this paper, we will correct these errors and construct a new integrable couplings of soliton hierarchy with self-consistent sources by making use of loop algebra $\widetilde{sl}(4)$.

This paper is organized as follows. In section Ⅱ, we will lead to a kind of integrable couplings of soliton hierarchies with self-consistent sources by making use of $\widetilde{sl}(4)$. In section Ⅲ, integrable couplings of the Li soliton hierarchy with self-consistent sources is derived by making use of loop algebra $\widetilde{sl}(4)$. In section Ⅳ, integrable couplings of the Broer-Kaup-Kupershmidt soliton hierarchy with self-consistent sources is derived by making use of loop algebra $\widetilde{sl}(4)$. Finally, some conclusions are given.

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II. A New Integrable Couplings of Soliton Hierarchy with Self-consistent Sources Associated with $\widetilde{sl}(4)$

In the following, we consider a set of matrix Lie algebra $sl(4)$[5

$$g_1=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},g_2=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$
$$g_3=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&1&1\\0&0&-1&0\end{pmatrix},g_4=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\0&0&0&0\\0&0&0&0\end{pmatrix},\tag{1}$$
$$g_5=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ g_6=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

It is easy to verify that

$$\begin{aligned}\mathrm{sl}(4)&=\mathrm{span}\,\{\mathrm{g}_1,\mathrm{g}_2,\mathrm{g}_3,\mathrm{g}_4,\mathrm{g}_5,\mathrm{g}_6\},\\ \mathrm{sl}(4)_1&=\mathrm{span}\,\{\mathrm{g}_1,\mathrm{g}_2,\mathrm{g}_3\},\\ \mathrm{sl}(4)_2&=\mathrm{span}\,\{\mathrm{g}_4,\mathrm{g}_5,\mathrm{g}_6\}\end{aligned}\tag{2}$$

Construct three Lie algebras, and satisfy

$$sl(4)=sl(4)_1\oplus sl(4)_2,$$
$$[sl(4)_1,sl(4)_2]=sl(4)_1sl(4)_2-sl(4)_2sl(4)_1\subseteq sl(4)_2.$$

The corresponding loop Lie algebra are defined as follows:

$$\begin{aligned}\widetilde{sl}(4)&=\{A|A\in R[\lambda,\lambda^{-1}]\otimes sl(4)\},\\ \widetilde{sl}(4)_1&=\{A|A\in R[\lambda,\lambda^{-1}]\otimes sl(4)_1\},\\ \widetilde{sl}(4)_2&=\{A|A\in R[\lambda,\lambda^{-1}]\otimes sl(4)_2\}.\end{aligned}$$

Consider the auxiliary linear problem

$$\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix}_x=U(u,\lambda)\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix},\tag{3a}$$

$$\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix}_{t_n}=V^{(n)}(u,\lambda)\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix},\tag{3b}$$

where

$U(u,\lambda)=g_1(\lambda)+\sum_{i=1}^{6}u_ig_i(\lambda)$, $u=(u_1,\ldots,u_6)^{\mathrm{T}}$, $u_i=u_i(x,t)(i=1,2,\ldots 6)$, $\phi_j=\phi_j(x,t)(\mathrm{j}=1,2,3,4)$ are field variables defining on $x\in R,\ t\in R, g_{\mathrm{i}}(\lambda)\in\widetilde{sl}(4)$, $\widetilde{sl}(4)$ denotes a infinite Lie algebra over R. When $\frac{\partial\lambda}{\partial t}=0$, the compatibility conditions of (3) gives rise to the well-known zero curvature equation

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0,\ n=1,2,\cdots,.\tag{4}$$

The general scheme of searching for the consistent $V^{(n)}$ and generating a hierarchy of soliton equations was proposed as follows[8]. First, we solve the stationary zero curvature equation

$$V_x=UV-VU,$$

$$V = \sum_{m=0}^{\infty} V_m(\mathrm{u})\lambda^{-m} =$$

$$\sum_{m=0}^{\infty}\begin{pmatrix} a_m & b_m+c_m & d_m & e_m+f_m \\ b_m-c_m & -a_m & e_m-f_m & -d_m \\ 0 & 0 & a_m & b_m+c_m \\ 0 & 0 & b_m-c_m & -a_m \end{pmatrix}\lambda^{-\mathrm{m}}. \quad (5)$$

Then we search for $\Delta_n \in \widetilde{sl}(4)$ such that $V^{(n)}$ can be constructed by

$$V^{(n)} = \sum_{m=0}^{n} V_m(\mathrm{u})\lambda^{n-m} + \Delta_n(u,\lambda),$$
$$\Delta_n(u,\lambda) = \Delta_{n1}g_1 + \Delta_{n2}g_2 + \Delta_{n3}g_3 + \Delta_{n4}g_4 + \Delta_{n5}g_5 + \Delta_{n6}g_6 \quad (6)$$

where Δ_{ni} are linear functions of a_n, b_n, c_n, d_n, e_n and f_n.

Under certain conditions, there is a constant γ such that the so-called variational identity holds[9].

$$\frac{\delta}{\delta u_i}\mathrm{Tr}\left(V\frac{\partial U}{\partial\lambda}\right) = \left(\lambda^{-\gamma}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\gamma}\right)\mathrm{Tr}\left(V\frac{\partial U}{\partial u_i}\right), \quad (7)$$

where Tr denotes the trace of a matrix. Define a scalar $H = H(u,\ \lambda)$ by the equation

$$\left(\lambda^{-\gamma}\left(\frac{\partial}{\partial\lambda}\right)\lambda^{\gamma}\right)H = \mathrm{Tr}\left(V\frac{\partial U}{\partial\lambda}\right),$$
$$H = \sum_{m=0}^{\infty} H_m(u)\lambda^{-m}. \quad (8)$$

From the variational identity (7), we obtain that

$$\sum_{m=0}^{\infty}\frac{\delta H_m}{\delta u_i}\lambda^{-m} = \mathrm{Tr}\left(V\frac{\partial U}{\partial u_i}\right) = \sum_{m=0}^{\infty}\mathrm{Tr}(V_m g_i)\lambda^{-m}. \quad (9)$$

The sets $\{H_{\mathrm{m}}\}$ proves the conserved densities of (4). In [8], Ma proposed the generalized Tu scheme to find a Hamiltonian function, a recursion operator L and symplectic operator J of the hierarchy based on stationary zero stationary equation (5). The Hamiltonian form with H_{n} can be written as

$$u_{t_n} = J\frac{\delta H_n}{\delta u},\ n = 1, 2, \ldots. \quad (10)$$
$$\frac{\delta H_n}{\delta u} = L\frac{\delta H_{n-1}}{\delta u} = \ldots = L^n\frac{\delta H_0}{\delta u},\ n = 1, 2, \ldots, \quad (11)$$

where $\frac{\delta}{\delta u} = (\frac{\delta}{\delta u_1}, \ldots, \frac{\delta}{\delta u_6})^T$.

We consider the auxiliary problem of (3). For N distinct λ_{j}, $j = 1, \ldots, N$, the following systems result from (3) as follows

$$\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\\ \phi_{4j}\end{pmatrix}_x = \left(g_1(\lambda_{\mathrm{j}}) + \sum_{\mathrm{i}=1}^{6} u_i g_i(\lambda_{\mathrm{j}})\right)\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\\ \phi_{4j}\end{pmatrix}, \quad (12a)$$

$$\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\\ \phi_{4j}\end{pmatrix}_{t_n} = \left(\sum_{m=0}^{\mathrm{n}} V_m(\mathrm{u})\lambda_{\mathrm{j}}^{n-m} + \Delta_n(u,\lambda_{\mathrm{j}})\right)\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\\ \phi_{4j}\end{pmatrix} \quad (12b)$$

The following equation is presented in Refs.[10, 11].

$$\frac{\delta H_k}{\delta u} + \sum_{\mathrm{j}=1}^{N}\alpha_{\mathrm{j}}\frac{\delta\lambda_{\mathrm{j}}}{\delta u} = 0, \quad (13)$$

where α_{j} are constants, $\frac{\delta H_k}{\delta u}$ determines a finite dimensional invariant sets for the flow (10).

For (12a), it is known (up to a constant factor) that

$$\frac{\delta\lambda_{\mathrm{j}}}{\delta u_i} = \frac{1}{2}\mathrm{Tr}\left(\Psi_j\frac{\partial U(u,\lambda_{\mathrm{j}})}{\partial u_i}\right) = \frac{1}{2}(\Psi_j g_i(\lambda_{\mathrm{j}})), \quad (14)$$

where $i = 1, 2, 3$. $j = 1, \ldots, N$ and

$$\Psi_j = \begin{pmatrix}\phi_{1\mathrm{j}}\phi_{2\mathrm{j}} & -\phi_{1\mathrm{j}}^2 & \phi_{3\mathrm{j}}\phi_{4\mathrm{j}} & -\phi_{3\mathrm{j}}^2\\ \phi_{2\mathrm{j}}^2 & -\phi_{1\mathrm{j}}\phi_{2\mathrm{j}} & \phi_{4\mathrm{j}}^2 & -\phi_{3\mathrm{j}}\phi_{4\mathrm{j}}\\ 0 & 0 & \phi_{1\mathrm{j}}\phi_{2\mathrm{j}} & -\phi_{1\mathrm{j}}^2\\ 0 & 0 & \phi_{2\mathrm{j}}^2 & -\phi_{1\mathrm{j}}\phi_{2\mathrm{j}}\end{pmatrix}.$$

For $\mathrm{u}_4, \mathrm{u}_5, \mathrm{u}_6$, if we set $g_i(\lambda_{\mathrm{j}}) = \begin{pmatrix}0 & A_{ij}\\ 0 & 0\end{pmatrix}$, $i = 4, 5, 6$. $\Psi_{j12} = \begin{pmatrix}\phi_{3j}\phi_{4j} & -\phi_{3j}^2\\ \phi_{4j}^2 & -\phi_{3j}\phi_{4j}\end{pmatrix}$, we demand that

$$\frac{\delta\lambda_{\mathrm{j}}}{\delta u_i} = \frac{1}{2}\mathrm{Tr}(\Psi_{j12}A_{ij}),\ \mathrm{i} = 4, 5, 6. \quad (15)$$

By using of (14) and (15), we present a kind of the integrable couplings of integrable soliton equation hierarchy with self-consistent sources

$$u_{t_n} = J\frac{\delta H_n}{\delta u} + J\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} = JL^n\frac{\delta H_0}{\delta u} + J\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}. \quad (16)$$

where $n = 1, 2, \ldots$.

Remark The representation of $\frac{\delta\lambda_{\mathrm{j}}}{\delta u_i}$ in [5]-[7] is wrong. In (14) and (15), we correct these errors.

III. The Integrable Couplings of Li Soliton Hierarchy With Self-Consistent Sources

In [12], Fan presented the zero curvature representation of Li soliton hierarchy. In this section,we consider the following enlarged Li matrix spectral problem

$$\phi_x = U(u,\lambda)\phi,$$
$$U(u,\lambda) = -g_1(1) + u_1(g_1(0) + g_3(0)) + u_2 g_2(0) + u_3(g_4(0) + g_6(0)) + u_4 g_5(0), \quad (17)$$

i.e.

$$U(u,\lambda) = \begin{pmatrix}-\lambda+u_1 & u_1+u_2 & u_3 & u_3+u_4\\ -u_1+u_2 & \lambda-u_1 & -u_3+u_4 & -u_3\\ 0 & 0 & -\lambda+u_1 & u_1+u_2\\ 0 & 0 & -u_1+u_2 & \lambda-u_1\end{pmatrix},$$

where λ is the spectral parameter.

To establish the integrable coupling system of the Lisoliton hierarchy, the stationary zero curvature equation $Vx = [U, V]$ of the spectral problem (17) through the generalized Tu scheme [8] is firstly solved. We assume that a solution of V is given by

$$V = \sum_{m=0}^{\infty} V_m(\mathrm{u})\lambda^{-m} = \sum_{\mathrm{m}=0}^{\infty}\lambda^{-m}\cdot$$
$$\begin{pmatrix} a_m & b_m+c_m & d_m & e_m+f_m \\ b_m-c_m & -a_m & e_m-f_m & -d_m \\ 0 & 0 & a_m & b_m+c_m \\ 0 & 0 & b_m-c_m & -a_m \end{pmatrix}. \quad (18)$$

Therefore, the condition (5) gives rise to the following recursion relations:

$$\begin{cases} a_{mx} = 2u_1 b_m - 2u_2 c_m,\\ b_{m+1} = -u_2 a_m + u_1 b_m - \frac{1}{2}c_{mx},\\ c_{m+1} = -u_1 a_m - \frac{1}{2}b_{mx} + u_1 c_m,\\ d_{mx} = 2u_3 b_m - 2u_4 c_m + 2u_1 e_m - 2u_2 f_m,\\ e_{m+1} = u_3 b_m - u_4 a_m - u_2 d_m + u_1 e_m - \frac{1}{2}f_{mx},\\ f_{m+1} = u_3 c_m - u_3 a_m - u_1 d_m - \frac{1}{2}e_{mx} + u_1 f_m. \end{cases} \quad (19)$$

If we set $a_0 = 2, b_0 = c_0 = d_0 = e_0 = f_0 = 0$, we see that all sets of functions $a_m, b_m, c_m, d_m, e_m, f_m$ are uniquely determined. In particular, the first few sets are:

$$\begin{cases} a_1 = 0, b_1 = -2u_2, c_1 = -2u_1, d_1 = 0,\\ e_1 = -2u_4, f_1 = -2u_3, a_2 = u_1^2 - u_2^2,\\ b_2 = u_{1x} - 2u_1u_2, c_2 = u_{2x} - 2u_1^2,\\ d_2 = 2u_1u_3 - 2u_3u_4,\\ e_2 = -2u_1u_4 - 2u_2u_3 + u_{3x},\\ f_2 = -4u_1u_3 + u_{4x}. \end{cases} \quad (20)$$

From (19), we can obtain the following recursion relations for $a_{m+1} - c_{m+1} + d_{m+1} - f_{m+1}, b_{m+1} + e_{m+1}$, $a_{m+1} - c_{m+1}$ and b_{m+1},

$$\begin{pmatrix} a_{m+1} - c_{m+1} + d_{m+1} - f_{m+1}\\ b_{m+1} + e_{m+1}\\ a_{m+1} - c_{m+1}\\ b_{m+1}\end{pmatrix} = L\begin{pmatrix} a_m - c_m + d_m - f_m\\ b_m + e_m\\ a_m - c_m\\ b_m\end{pmatrix} \quad (21)$$

where

$$L = \begin{pmatrix}\frac{1}{2}\partial - \partial^{-1}u_1\partial & L_{12} & \partial^{-1}u_3\partial & L_{14}\\ \frac{1}{2} & 0 & 0 & -u_3\\ 0 & 0 & \partial^{-1}u_1\partial + u_1 & L_{34}\\ 0 & 0 & \frac{1}{2}\partial - u_2 & 0\end{pmatrix},$$

where $L_{12} = -\partial^{-1}u_2\partial - u_2, L_{13} = u_3+,\ L_{14} = \partial^{-1}u_4\partial,\ L_{34} = -\partial^{-1}u_2\partial - u_2$.

Considering

$$V^{(n)}=\sum_{m=0}^{n}V_m(u)\lambda^{n-m}+\Delta_n,$$

$$\Delta_n=\begin{pmatrix}-a_n+c_n & 0 & -d_n+f_n & 0\\ 0 & a_n-c_n & 0 & d_n-f_n\\ 0 & 0 & -a_n+c_n & 0\\ 0 & 0 & 0 & a_n-c_n\end{pmatrix}. \quad (22)$$

from the zero curvature equation $U_t-V_x^{(n)}+[U,V^{(n)}]=0$,

we obtain a new coupling system

$$U_t-\left(\sum_{m=0}^{n}V_m(u)\lambda^{n-m}\right)_x+\left[U,\sum_{m=0}^{n}V_m(u)\lambda^{n-m}\right]-\Delta_{nx}+[U,\Delta_n]=0. \quad (23)$$

Substituting (22) into (23), we have

$$V_x^{(n)}-[U,V^{(n)}]=(-a_{nx}+c_{nx})g_1+b_{nx}g_2+(-a_{nx}+c_{nx})g_3+(-d_{nx}+f_{nx})g_4+e_{nx}g_5+(-d_{nx}+f_{nx})g_6.$$

Thus the zero curvature equation (23) determines the following system

$$u_t=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}_t=\begin{pmatrix}0&0&-\partial&0\\0&0&0&\partial\\-\partial&0&\partial&0\\0&\partial&0&-\partial\end{pmatrix}\begin{pmatrix}a_n-c_n+d_n-f_n\\b_n+e_n\\a_n-c_n\\b_n\end{pmatrix}$$

$$=J\begin{pmatrix}a_n-c_n+d_n-f_n\\b_n+e_n\\a_n-c_n\\b_n\end{pmatrix}=JL^n\begin{pmatrix}2\\0\\2\\0\end{pmatrix}. \quad (24)$$

In order to construct non-degenerate bilinear forms, we transform the algebra $sl(4)$ defined by (1) into a vector form.

Define the mapping:

$$\delta: sl(4)\mapsto R^6,$$
$$A\mapsto(a_1,a_2,a_3,a_4,a_5,a_6)^T,$$

where $A=a_1g_1+a_2g_2+\cdots+a_6g_6\in sl(4)$.

This mapping δ induces a Lie algebra on R^6 isomorphic to the matrix Lie algebra sl(4). $\forall\, a=(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6)^T$, $b=(b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6)^T\in R^6$, a bilinear form can be defined by

$$\langle a,b\rangle=a^TFb, \quad (25)$$

where F is a constant matrix. As

$$(\delta[A,\ B])^T=a^TFb=a^TR(b), \quad (26)$$

where

$$R(b)=\begin{pmatrix}0&2b_3&2b_2&0&2b_6&2b_5\\-2b_3&0&-2b_1&-2b_6&0&-2b_4\\2b_2&-2b_1&0&2b_5&-2b_4&0\\0&0&0&0&2b_3&2b_2\\0&0&0&-2b_3&0&-2b_1\\0&0&0&2b_2&-2b_1&0\end{pmatrix}.$$

According to the symmetry and invariance of the bilinear form, so F and $R(b)$ must satisfy $F^T=F,\ R(b)F=-(R(b)F)^T$. So we determine the symmetric matrix F:

$$F=\begin{pmatrix}1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&-1&0&0&-1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&-1&0&0&0\end{pmatrix}.$$

We construct the Hamiltonian structure of the Li hierarchy by using of the variational identity [10].

$$\frac{\delta}{\delta u_i}\langle V_n,\frac{\partial U}{\partial\lambda}\rangle=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\langle V_n,\frac{\partial U}{\partial u_i}\rangle. \quad (27)$$

Through a direct computation, we obtain

$$\langle V_n,\frac{\partial U}{\partial\lambda}\rangle=-a_n-d_n,\quad \langle V_n,\frac{\partial U}{\partial u_1}\rangle=a_n-c_n+d_n-f_n,$$
$$\langle V_n,\frac{\partial U}{\partial u_2}\rangle=b_n+e_n,$$
$$\langle V_n,\frac{\partial U}{\partial u_3}\rangle=a_n-c_n,$$
$$\langle V_n,\frac{\partial U}{\partial u_4}\rangle=b_n.$$

Substituting the above formulas into the above identity identity (27), it give s rise to

$$\frac{\delta}{\delta u}\langle V,\frac{\partial U}{\partial\lambda}\rangle=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}\langle V,\frac{\partial U}{\partial u_1}\rangle\\\langle V,\frac{\partial U}{\partial u_2}\rangle\\\langle V,\frac{\partial U}{\partial u_3}\rangle\\\langle V,\frac{\partial U}{\partial u_4}\rangle\end{pmatrix}, \quad (28)$$

Comparing the coefficient of λ^{-n-1}, we get

$$\left(\frac{\delta}{\delta u_1},\frac{\delta}{\delta u_2},\frac{\delta}{\delta u_3},\frac{\delta}{\delta u_4}\right)(-a_{n+1}-d_{n+1})$$
$$=(\gamma-n)(a_n-c_n+d_n-f_n,\ b_n+e_n,a_n-c_n,b_n). \quad (29)$$

To fix the constant γ, we simplify set $n=1$ in (29) and find that $\gamma=0$. Therefore, we conclude that

$$G_n=\frac{\delta H_n}{\delta u},H_n=\int\frac{a_{n+1}+d_{n+1}}{n}dx,\ n\geq 1. \quad (30)$$

Hence, the Hamiltonian structure of the coupling Li hierarchy is constructed as follows

$$u_t=JG_n=J\frac{\delta H_n}{\delta u}. \quad (31)$$

Next, we will construct the integrable couplings of Li equations with self-consistent sources. Considering the auxiliary linear problems

$$=U(u,\lambda_j)\begin{pmatrix}\phi_{1j}\\\phi_{2j}\\\phi_{3j}\\\phi_{4j}\end{pmatrix},$$

$$\begin{pmatrix}\phi_{1j}\\\phi_{2j}\\\phi_{3j}\\\phi_{4j}\end{pmatrix}_t=V^{(n)}(u,\lambda_j)\begin{pmatrix}\phi_{1j}\\\phi_{2j}\\\phi_{3j}\\\phi_{4j}\end{pmatrix}. \quad (32)$$

From (14) and (15), we can obtain

$$\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}=\begin{pmatrix}\langle\Phi_1,\Phi_2\rangle+\frac{1}{2}(\langle\Phi_1,\Phi_1\rangle+\langle\Phi_2,\Phi_2\rangle)\\\frac{1}{2}(\langle\Phi_2,\Phi_2\rangle-\langle\Phi_1,\Phi_1\rangle)\\\langle\Phi_3,\Phi_4\rangle+\frac{1}{2}(\langle\Phi_3,\Phi_3\rangle+\langle\Phi_4,\Phi_4\rangle)\\\frac{1}{2}(\langle\Phi_4,\Phi_4\rangle-\langle\Phi_3,\Phi_3\rangle)\end{pmatrix} \quad (33)$$

where $\Phi_i=(\phi_{i1},\cdots,\phi_{iN})^T$, i = 1, 2, 3, 4. $\langle\cdot,\cdot\rangle$ denotes the inner product in R^N.

By using the results in [10, 11], we have the following equation

$$\frac{\delta H_k}{\delta u}+\sum_{j=1}^{N}\alpha_j\frac{\delta\lambda_j}{\delta u}=0, \quad (34)$$

where α_j are constants, $\frac{\delta H_k}{\delta u}$ determines a finite dimensional invariant sets for the flow $u_{t_n}=J\frac{\delta H_n}{\delta u}$, n = 1,2, ….

From (16), we obtain a kind of integrable couplings of Li soliton equation hierarchy with self-consistent sources as follows

$$u_t=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}_t=J\begin{pmatrix}a_n-c_n+d_n-f_n\\b_n+e_n\\a_n-c_n\\b_n\end{pmatrix}$$
$$-\frac{1}{2}J\begin{pmatrix}-2\langle\Phi_1,\Phi_2\rangle-\langle\Phi_1,\Phi_1\rangle-\langle\Phi_2,\Phi_2\rangle\\\langle\Phi_1,\Phi_1\rangle-\langle\Phi_2,\Phi_2\rangle\\-2\langle\Phi_3,\Phi_4\rangle-\langle\Phi_3,\Phi_3\rangle-\langle\Phi_4,\Phi_4\rangle\\\langle\Phi_3,\Phi_3\rangle-\langle\Phi_4,\Phi_4\rangle\end{pmatrix}. \quad (35)$$

A novel integrable coupling system of the Li soliton equation hierarchy with self-consistent sources is presented as follows :

When n =1, we have

$$u_{1t}=-2u_{1x}-\sum_{j=1}^{N}((\phi_{3j}\phi_{4j})_x+\phi_{3j}\phi_{3jx}+\phi_{4j}\phi_{4jx}),$$
$$u_{2t}=-2u_{2x}+\sum_{j=1}^{N}(\phi_{4j}\phi_{4jx}-\phi_{3j}\phi_{3jx}),$$
$$u_{3t}=-2u_{3x}+\sum_{j=1}^{N}(\phi_{3j}\phi_{3jx}+\phi_{4j}\phi_{4jx}+(\phi_{3j}\phi_{4j})_x)$$
$$-\sum_{j=1}^{N}(\phi_{1j}\phi_{1jx}+\phi_{2j}\phi_{2jx}+(\phi_{1j}\phi_{2j})_x),$$

$$u_{4t} = -2u_{4x} + \sum_{j=1}^{N}(\phi_{2j}\phi_{2jx} + \phi_{3j}\phi_{3jx} - \phi_{1j}\phi_{1jx} - \phi_{4j}\phi_{4jx}),$$
$$\phi_{1jx} = (-\lambda_j + u_1)\phi_{1j} + (u_1 + u_2)\phi_{2j} + u_3\phi_{3j} + (u_3 + u_4)\phi_{4j},$$
$$\phi_{2jx} = (u_2 - u_1)\phi_{1j} + (\lambda_j - u_1)\phi_{2j} + (u_4 - u_3)\phi_{3j} - u_3\phi_{4j},$$
$$\phi_{3jx} = (-\lambda_j + u_1)\phi_{3j} + (u_1 + u_2)\phi_{4j},$$
$$\phi_{4jx} = (u_2 - u_1)\phi_{3j} + (\lambda_j - u_1)\phi_{4j},\ j = 1, \cdots, N \quad (36)$$

When $n = 2$, we have

$$u_{1t} = u_{2xx} + 2u_2u_{2x} - 6u_1u_{1x} - \sum_{j=1}^{N}((\phi_{3j}\phi_{4j})_x + \phi_{3j}\phi_{3jx} + \phi_{4j}\phi_{4jx}),$$
$$u_{2t} = u_{1xx} - 2(u_1u_2)_x + \textstyle\sum_{j=1}^{N}(\phi_{4j}\phi_{4jx} - \phi_{3j}\phi_{3jx}),$$
$$u_{3t} = u_{4xx} - 6(u_1u_3)_x + 2(u_2u_4)_x + \sum_{j=1}^{N}(\phi_{3j}\phi_{3jx} + \phi_{4j}\phi_{4jx} + (\phi_{3j}\phi_{4j})_x) - \sum_{j=1}^{N}(\phi_{1j}\phi_{1jx} + \phi_{2j}\phi_{2jx} + (\phi_{1j}\phi_{2j})_x),$$
$$u_{4t} = u_{3xx} - 2(u_1u_4)_x - 2(u_2u_3)_x + \textstyle\sum_{j=1}^{N}(\phi_{2j}\phi_{2jx} + \phi_{3j}\phi_{3jx} - \phi_{1j}\phi_{1jx} - \phi_{4j}\phi_{4jx}),$$
$$\phi_{1jx} = (-\lambda_j + u_1)\phi_{1j} + (u_1 + u_2)\phi_{2j} + u_3\phi_{3j} + (u_3 + u_4)\phi_{4j},$$
$$\phi_{2jx} = (u_2 - u_1)\phi_{1j} + (\lambda_j - u_1)\phi_{2j} + (u_4 - u_3)\phi_{3j} - u_3\phi_{4j},$$
$$\phi_{3jx} = (-\lambda_j + u_1)\phi_{3j} + (u_1 + u_2)\phi_{4j},$$
$$\phi_{4jx} = (u_2 - u_1)\phi_{3j} + (\lambda_j - u_1)\phi_{4j},\ j = 1, \cdots, N. \quad (37)$$

We obtain the integrable coupling of Li equation hierarchy with self-consistent sources. When $u_3 = u_4 = 0$, (7) can be reduced to the Li equation hierarchy. When $n = 2$, $u_1 = u_2 = u$, $u_3 = u_4 = 0$, (37) can be reduced to Burgers equation[12] with self-consistent sources

$$u_t = u_{xx} - 4uu_{xx} + \sum_{j=1}^{N}(\phi_{2j}\phi_{2jx} - \phi_{1j}\phi_{1jx}), = (-\lambda_j + u)\phi_{1j} + 2u\phi_{2j},$$
$$\phi_{2jx} = (\lambda_j - u)\phi_{2j},\ j = 1, \cdots, N. \quad (38)$$

IV. The Integrable Couplings of BKK Soliton Hierarchy With Self-Consistent sources

The BKK hierarchy was brought to prominence by Kupershmidt in [13], where it was attributed to Broer and Kaup. It seems, however, that it should also be attributed to Whitham.. It is frequently just referred to as a “Boussinesq-type” hierarchy. The [14] considers the inverse spectral problem related to the BKK system by some transformation, and the [13] considers the algebraic structure of the BKK system in detail. In this section, we consider the following enlarged BKK matrix spectral problem

$$\phi_x = U(u,\lambda)\phi,$$
$$U(u,\lambda) = g_1(1) + u_1g_1(0) + \tfrac{1}{2}u_2(g_2(0) + g_3(0)) + \tfrac{1}{2}(g_2(0) - g_3(0)) + u_3g_4(0) + \tfrac{1}{2}u_4(g_5(0) + g_6(0)), \quad (39)$$

i.e.

$$U(u,\lambda) = \begin{pmatrix} \lambda + u_1 & u_2 & u_3 & u_4 \\ 1 & -\lambda - u_1 & 0 & -u_3 \\ 0 & 0 & \lambda + u_1 & u_2 \\ 0 & 0 & 0 & -\lambda - u_1 \end{pmatrix},$$

where λ is the spectral parameter.

To establish the integrable couplings system of the BKK soliton hierarchy, the stationary zero curvature equation $Vx = [U, V]$ of the spectral problem (39) through the generalized Tu scheme [8] is firstly solved. We assume that a solution of V is given by

$$V = \sum_{m=0}^{\infty} V_m(\mathrm{u})\lambda^{-m} = \textstyle\sum_{m=0}^{\infty}\lambda^{-m} \cdot \begin{pmatrix} a_m & b_m & d_m & e_m \\ c_m & -a_m & f_m & -d_m \\ 0 & 0 & a_m & b_m \\ 0 & 0 & c_m & -a_m \end{pmatrix}. \quad (40)$$

Therefore, the condition (5) gives rise to the following recursion relations:

$$\begin{cases} a_{mx} = -b_m + u_2c_m, \\ b_{m+1} = u_2a_m - u_1b_m + \frac{1}{2}b_{mx}, \\ c_{m+1} = a_m - \frac{1}{2}c_{mx} - u_1c_m, \\ d_{mx} = u_4c_m - e_m + u_2f_m, \\ e_{m+1} = -u_3b_m + u_4a_m + u_2d_m - u_1e_m + \frac{1}{2}e_{mx}, \\ f_{m+1} = -u_3c_m + d_m - \frac{1}{2}f_{mx} - u_1f_m. \end{cases} \quad (41)$$

If we set $a_0 = 1$, $d_0 = 1$, $b_0 = c_0 = e_0 = f_0 = 0$, we see that all sets of functions $a_m, b_m, c_m, d_m, e_m, f_m$ are uniquely determined. In particular, the first few sets are:

$$\begin{pmatrix} a_1 = 0, b_1 = u_2, c_1 = 1, d_1 = 0, e_1 = u_2 + u_4, \\ f_1 = 1, a_2 = -\frac{1}{2}u_2, b_2 = \frac{1}{2}u_{2x} - u_1u_2, \\ c_2 = -u_1, d_2 = -\frac{1}{2}(u_2 + u_4), \\ e_2 = \frac{1}{2}(u_2 + u_4)_x - u_1(u_2 + u_4) - u_2u_3, \\ f_2 = -(u_1 + u_3). \end{pmatrix} \quad (42)$$

From (41), we can obtain the following recursion relations for $2a_{m+1} + 2d_{m+1}, c_{m+1} + f_{m+1}$, $2a_{m+1}$ and c_{m+1},

$$\begin{pmatrix} 2a_{m+1} + 2d_{m+1} \\ c_{m+1} + f_{m+1} \\ 2a_{m+1} \\ c_{m+1} \end{pmatrix} = L \begin{pmatrix} 2a_m + 2d_m \\ c_m + f_m \\ 2a_m \\ c_m \end{pmatrix} \quad (43)$$

wth

$$L = \begin{pmatrix} \frac{1}{2}\partial - \partial^{-1}u_1\partial & L_{12} & -\partial^{-1}u_3\partial & L_{14} \\ \frac{1}{2} & L_{22} & 0 & -u_3 \\ 0 & 0 & \frac{1}{2}\partial - \partial^{-1}u_1\partial & L_{34} \\ 0 & 0 & \frac{1}{2} & L_{44} \end{pmatrix},$$

where $L_{12} = -\partial^{-1}u_2\partial - u_2, L_{22} = -\frac{1}{2}\partial - u_1$, $L_{14} = -\partial^{-1}u_4\partial - u_4$, $L_{34} = -\partial^{-1}u_2\partial - u_2, L_{44} = -\frac{1}{2}\partial - u_1$.

Considering

$$V^{(n)} = \sum_{m=0}^{n} V_m(u)\lambda^{n-m} + \Delta_n,$$
$$\Delta_n = \begin{pmatrix} -c_{n+1} & 0 & -f_{n+1} & 0 \\ 0 & c_{n+1} & 0 & f_{n+1} \\ 0 & 0 & -c_{n+1} & 0 \\ 0 & 0 & 0 & c_{n+1} \end{pmatrix}. \quad (44)$$

From the zero curvature equation $U_t - V_x^{(n)} + [U, V^{(n)}] = 0$, we obtain a new coupling system

$$U_t - (\textstyle\sum_{m=0}^{n} V_m(u)\lambda^{n-m})_x + [U, \sum_{m=0}^{n} V_m(u)\lambda^{n-m}] - \Delta_{nx} + [U, \Delta_n] = 0. \quad (45)$$

Substituting (44) into (45), we have

$$V_x^{(n)} - [U, V^{(n)}] = -c_{n+1,x}g_1 - a_{n+1,x}(g_2 + g_3) + -f_{n+1,x}g_4 - d_{n+1,x}(g_5 + g_6).$$

Thus the zero curvature equation (45) determines the following system

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & 0 & \partial \\ 0 & 0 & \partial & 0 \\ 0 & \partial & 0 & -\partial \\ \partial & \partial & -\partial & 0 \end{pmatrix} \begin{pmatrix} -2(a_{n+1} + d_{n+1}) \\ -(c_{n+1} + f_{n+1}) \\ -2a_n \\ -c_n \end{pmatrix}$$

$$= \begin{pmatrix} -2(a_{n+1} + d_{n+1}) \\ -(c_{n+1} + f_{n+1}) \\ -2a_n \\ -c_n \end{pmatrix} J. \quad (46)$$

In order to construct non-degenerate bilinear forms, we transform the algebra $sl(4)$ defined by (1) into a vector form.

Define the mapping:

$$\delta: sl(4) \mapsto R^6,$$
$$A \mapsto (a_1, a_2, a_3, a_4, a_5, a_6)^T,$$

where $A = a_1 g_1 + \frac{1}{2}(a_2 + a_3) g_2 + \frac{1}{2}(a_2 - a_3) g_3 + a_4 g_4 + \frac{1}{2}(a_5 + a_6) g_5 + \frac{1}{2}(a_5 - a_6) g_6 \in sl(4)$.

This mapping δ induces a Lie algebra on R^6 isomorphic to the matrix Lie algebra sl(4). $\forall a = (a_1, a_2, a_3, a_4, a_5, a_6)^T$, $b = (b_1, b_2, b_3, b_4, b_5, b_6)^T \in R^6$, a bilinear form can be defined by

$$\langle a, b \rangle = a^T F b, \tag{47}$$

where F is a constant matrix. As

$$(\delta[A, B])^T = a^T F b = a^T R(b), \tag{48}$$

where

$$R(b) = \begin{pmatrix} 0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 \\ b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0 \\ -b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 \\ 0 & 0 & 0 & 0 & 2b_2 & -2b_3 \\ 0 & 0 & 0 & b_3 & -2b_1 & 0 \\ 0 & 0 & 0 & -b_2 & 0 & 2b_1 \end{pmatrix}.$$

According to the symmetry and invariance of the bilinear form, so F and $R(b)$ must satisfy $F^T = F$, $R(b)F = -(R(b)F)^T$. So we determine the symmetric matrix F:

$$F = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We construct the Hamiltonian structure of the BKK hierarchy by using of the variational identity [10].

$$\frac{\delta}{\delta u_i}\langle V_n, \frac{\partial U}{\partial \lambda}\rangle = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\langle V_n, \frac{\partial U}{\partial u_i}\rangle. \tag{49}$$

Through a direct computation, we obtain

$$\langle V_n, \frac{\partial U}{\partial \lambda}\rangle = 2a_n + 2d_n,$$
$$\langle V_n, \frac{\partial U}{\partial u_1}\rangle = 2a_n + 2d_n, \langle V_n, \frac{\partial U}{\partial u_2}\rangle = c_n + f_n,$$
$$\langle V_n, \frac{\partial U}{\partial u_3}\rangle = 2a_n, \quad \langle V_n, \frac{\partial U}{\partial u_4}\rangle = c_n.$$

Substituting the above formulas into the above identity identity (49), it give s rise to

$$\frac{\delta}{\delta u}\langle V, \frac{\partial U}{\partial \lambda}\rangle = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix} \langle V, \frac{\partial U}{\partial u_1}\rangle \\ \langle V, \frac{\partial U}{\partial u_2}\rangle \\ \langle V, \frac{\partial U}{\partial u_3}\rangle \\ \langle V, \frac{\partial U}{\partial u_4}\rangle \end{pmatrix}, \tag{50}$$

Comparing the coefficient of λ^{-n-1}, we get

$$\left(\frac{\delta}{\delta u_1}, \frac{\delta}{\delta u_2}, \frac{\delta}{\delta u_3}, \frac{\delta}{\delta u_4}\right)(2a_{n+1} + 2d_{n+1})$$
$$= (\gamma - n)(2a_n + 2d_n, c_n + f_n, 2a_n, c_n). \tag{51}$$

To fix the constant γ, we simplify set $n = 1$ in (51) and find that $\gamma = 0$. Therefore, we conclude that

$$G_n = \frac{\delta H_n}{\delta u}, H_n = 2\int \frac{a_{n+2} + d_{n+2}}{n+1} dx, \ n \geq 0. \tag{52}$$

Hence, the Hamiltonian structure of the coupling BKK hierarchy is constructed as follows

$$u_t = J G_n = J\frac{\delta H_n}{\delta u}. \tag{53}$$

Next, we will construct the integrable couplings of BKK equations with self-consistent sources. Considering the auxiliary linear problems

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \\ \phi_{4j} \end{pmatrix}_x = U(u, \lambda_j)\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \\ \phi_{4j} \end{pmatrix},$$
$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \\ \phi_{4j} \end{pmatrix}_t = V^{(n)}(u, \lambda_j)\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \\ \phi_{4j} \end{pmatrix}. \tag{54}$$

From (14) and (15), we can obtain

$$\sum_{j=1}^{N}\frac{\delta \lambda_j}{\delta u} = \begin{pmatrix} \langle \Phi_1, \Phi_2\rangle \\ \frac{1}{2}\langle \Phi_2, \Phi_2\rangle \\ \langle \Phi_3, \Phi_4\rangle \\ \frac{1}{2}\langle \Phi_4, \Phi_4\rangle \end{pmatrix}, \tag{55}$$

where $\Phi_i = (\phi_{i1}, \cdots, \phi_{iN})^T$, i = 1, 2, 3, 4. $\langle \cdot, \cdot \rangle$ denotes the inner product in R^N.

By using the results in [10, 11], we have the following equation

$$\frac{\delta H_k}{\delta u} + \sum_{j=1}^{N}\alpha_j \frac{\delta \lambda_j}{\delta u} = 0, \tag{56}$$

where α_j are constants, $\frac{\delta H_k}{\delta u}$ determines a finite dimensional invariant sets for the flow $u_{t_n} = J\frac{\delta H_n}{\delta u}$, $n = 1, 2, \ldots$.

From (16), we obtain a kind of integrable couplings of BKK soliton equation hierarchy with self-consistent sources as follows

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = J\begin{pmatrix} 2a_n + 2d_n \\ c_n + f_n \\ 2a_n \\ c_n \end{pmatrix} + \frac{1}{2}J\begin{pmatrix} 2\langle \Phi_1, \Phi_2\rangle \\ \langle \Phi_2, \Phi_2\rangle \\ 2\langle \Phi_3, \Phi_4\rangle \\ \langle \Phi_4, \Phi_4\rangle \end{pmatrix} \tag{57}$$

A novel integrable coupling system of the BKK soliton equation hierarchy with self-consistent sources is presented as follows :

when $n = 1$, we have

$$u_{1t} = u_{1x} + \sum_{j=1}^{N}\phi_{4j}\phi_{4jx},$$
$$u_{2t} = u_{2x} + \sum_{j=1}^{N}(\phi_{3jx}\phi_{4j} + \phi_{3j}\phi_{4jx}),$$
$$u_{3t} = u_{1x} + u_{3x} + \sum_{j=1}^{N}(\phi_{2j}\phi_{2jx} - \phi_{4j}\phi_{4jx})$$
$$u_{4t} = u_{2x} + u_{4x} + \sum_{j=1}^{N}(\phi_{1jx}\phi_{2j} + \phi_{1j}\phi_{2jx} - \phi_{3jx}\phi_{4j} - \phi_{3j}\phi_{4jx}),$$
$$\phi_{1jx} = (\lambda_j + u_1)\phi_{1j} + u_2\phi_{2j} + u_3\phi_{3j} + u_4\phi_{4j},$$
$$\phi_{2jx} = \phi_{1j} - (\lambda_j + u_1)\phi_{2j} - u_3\phi_{4j},$$
$$\phi_{3jx} = (\lambda_j + u_1)\phi_{3j} + u_2\phi_{4j},$$
$$\phi_{4jx} = \phi_{3j} - (\lambda_j + u_1)\phi_{4j}, j = 1, \cdots, N. \tag{58}$$

when $n = 2$, we have

$$u_{1t} = -\frac{1}{2}u_{1xx} - 2u_1u_{1x} + \frac{1}{2}u_{2x} - \sum_{j=1}^{N}\phi_{4j}\phi_{4jx},$$
$$u_{2t} = \frac{1}{2}u_{2xx} - 2(u_1u_2)_x + \sum_{j=1}^{N}(\phi_{3jx}\phi_{4j} + \phi_{3j}\phi_{4jx}),$$
$$u_{3t} = -\frac{1}{2}u_{3xx} - 2(u_1u_3)_x - 2u_1u_{1x} - \frac{1}{2}u_{1xx} + \frac{1}{2}u_{2x} + \frac{1}{2}u_{4x} + \sum_{j=1}^{N}(\phi_{2j}\phi_{2jx} - \phi_{4j}\phi_{4jx})$$
$$u_{4t} = \frac{1}{2}u_{2xx} + \frac{1}{2}u_{4xx} - 2(u_1u_2)_x - 2(u_1u_4)_x - 2(u_2u_3)_x + \sum_{j=1}^{N}(\phi_{1jx}\phi_{2j} + \phi_{1j}\phi_{2jx} - \phi_{3jx}\phi_{4j} - \phi_{3j}\phi_{4jx}),$$
$$\phi_{1jx} = (\lambda_j + u_1)\phi_{1j} + u_2\phi_{2j} + u_3\phi_{3j} + u_4\phi_{4j},$$
$$\phi_{2jx} = \phi_{1j} - (\lambda_j + u_1)\phi_{2j} - u_3\phi_{4j},$$
$$\phi_{3jx} = (\lambda_j + u_1)\phi_{3j} + u_2\phi_{4j},$$
$$\phi_{4jx} = \phi_{3j} - (\lambda_j + u_1)\phi_{4j}, j = 1, \cdots, N. \tag{59}$$

V. Conclusions

In this paper, we have obtained a new integrable couplings of soliton equations hierarchy with self-consistent sources based on loop algebras $\widetilde{sl}(4)$. In fact, integrable couplings of soliton hierarchy with self-consistent sources in [5-7] were obtained by our method. Our method can be applied to other integrable couplings of soliton hierarchies with self-consistent sources.

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